STATISTICAL PROPERTIES OF THE EIGENFUNCTIONS ON QUANTUM GRAPHS

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September 2009

A DISSERTATION SUBMITTED TO THE UNIVERSITY OF BRISTOL IN ACCORDANCE WITH THE REQUIREMENTS OF THE DEGREE OF DOCTOR OF PHILOSOPHY IN THE FACULTY OF SCIENCE

To the memory of my mother.

Abstract

We investigate the full spatial statistics of the energy eigenfunctions on quantum graphs. The autocorrelation functions obtained for an individual quantum graph exhibit an universal component, which completely determines a Gaussian random waves model, and a system-dependent deviation. This deviation turns out to only depend on the graph through its underlying classical dynamics. Classical criteria for quantum universality to be asymptotically met in the large graphs limit are then extracted from these formulae. We use an exact field theoretic expression in term of a variant of a supersymmetric σ model, and a saddle-point analysis of this expression leads to the estimates. In particular, the second order autocorrelation functions are used to discuss the possible equidistribution of the energy eigenfunctions in the large graphs limit. When equidistribution is asymptotically realized, our theory predicts a rate of convergence that is a significant refinement of previous estimates, long assumed to be valid for quantum chaotic systems, agreeing with them in some situations but not all. The universal and system-dependent components of the second order autocorrelation functions are recovered by means of two approximated trace formulae, drawing in this way a parallel between our field theory and semiclassics.

Acknowledgements

The first person I would like to thank is Jon Keating, who has been a great advisor throughout these four years in Bristol. His ability to come up with bright new ideas every time I found myself in a dead end showed me the most creative and interesting side of research. Not only his scientific contribution has been very precious, but he is also the person who always cared, advised me, and so often understood and met my needs.

I also want to thank Sven Gnutzmann for having introduced me to the supersymmetry techniques, and for all the great times we had in Bristol, Cambridge, Nottingham, Rehovot, and Wittenberg. The very enjoyable discussions we had during my frequent week-long visits in Nottingham always led to a deeper understanding and to a big new step in the work related here.

Working in the Quantum Chaos Group in Bristol has been a very unique experience. The enthusiasm of all the members and secretaries, the high quality seminars and the very interactive and friendly atmosphere make it a great place to work. Henrick and Panos ensured the good ambience on the fourth floor of Howard House, and our climbing or pool evenings were the best way to relax after a working day.

The first three years of my Ph.D. studies have been funded by the Overseas Research Students Awards Scheme and the University of Bristol, and I am deeply indebted to them.

I also acknowledge the hospitality of the Weizmann Institute of Science, where this thesis has been finished. I am particularly grateful to Uzy Smilansky for having arranged and rearranged my visit in Israel when I had other

preoccupations in mind. I also enjoyed many enlightening discussions with him during which he pointed several relevant pre-existing works.

Finally, I address my special thanks to my family, Nicole, Georges and Elsa, to my grand-parents, and to Désirée, who have constantly been extremely supportive. Despite my numerous explanation attempts, they had no idea of what I was really working on, and they probably secretly imagined me working as a giant calculator, multiplying a lot of numbers together very fast all day. However, they have always valued my choices, and this is a great feeling. I realize that this thesis was as important to my mother as it is to me, and finishing it is probably a bigger relief for Désirée than for anyone else.

Author's Declaration

I declare that the work in this thesis was carried out in accordance with the Regulations of the University of Bristol. The work is original except where indicated by special reference in the text and no part of the dissertation has been submitted for any other degree. Any views expressed in the dissertation are those of the author and do not necessarily represent those of the University of Bristol. The thesis has not been presented to any other university for examination either in the United Kingdom or overseas.

Fabien Piotet

Date: January 2010

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Chapter 1

Quantum Graphs

1.1 Why Quantum Graphs?

Since the seminal paper [54] by T. Kottos and U. Smilansky in 1997, quantum graphs have become a paradigm for quantum chaos and universal interference effects which can be observed in the statistics of their spectra and of their eigenfunctions. T. Kottos and U. Smilansky numerically showed that the spectral statistics for a large class of quantum graphs follow the predictions of the Gaussian random matrix ensembles up to finite-size corrections. In 1999 and in 2001, G. Berkolaiko, E.B. Bogomolny and J.P. Keating found in [10] and [11] that some special graphs, namely, star graphs, exhibit intermediate spectral statistics which do not converge to any of the random matrix expectations. This discovery revealed the richness of quantum physics on graphs, and it prompted the question as to when the statistical properties of the spectrum are describable by means of random matrix theory. In 2004, S. Gnutzmann and A. Altland answered this question in [40] and [41] by means of a nonlinear supersymmetric σ model. These works provide a relatively complete understanding of the eigenvalue statistics of quantum graphs.

Until 2008, substantially less was known about the eigenfunctions on quantum graphs. The eigenfunctions are however of major interest in quantum chaos, since they are believed to distinguish between regular and irregular

classical motions in a more subtle way than the eigenvalues do. They are mainly supported on classical invariant structures [32] such as tori in classically integrable systems [59] and are believed to be generically spread over the configuration space in ergodic systems [62]. Moreover, in case of chaos, the generic eigenfunctions have been conjectured to behave like Gaussian random waves in the bulk of the system [16]. Hence, eigenfunction statistics enable to distinguish between different types of underlying classical mechanics. But the eigenfunctions also exhibit thinner properties of the dynamics when considered individually. For example, in chaotic systems, some eigenfunctions are strongly enhanced in the vicinity of unstable periodic orbits [46], a phenomenon known as scarring. This phenomenon has been the object of many different works such as [26], [68], [17], [63], [1], [37], [52], [72], [51], [49] and [35]. The scars are different from the eigenfunctions living around stable periodic orbits in integrable systems [22, 23], or around marginally stable orbits in mixing systems [58], in which case they are known as bouncing ball modes. In [50] and [60], some quantum graphs are proved to contain scars.

In [16], M.V. Berry calculated the spatial two-point autocorrelations of a typical energy eigenfunction ψ_n of a chaotic system in the semiclassical limit. For a planar billiard $\Omega \subset \mathbb{R}^2$, such a function satisfies the Helmholtz equation $\Delta\psi_n = -e_n\psi_n$ on the interior of Ω and Dirichlet boundary conditions on $\partial\Omega$, that is $\psi_n|_{\partial\Omega} = 0$. If the energy e_n is written $e_n = k_n^2$ for some positive wavenumber k_n, Berry's result reads

$$R_n(\boldsymbol{r}_1, \boldsymbol{r}_2) \equiv \frac{1}{|S|}\int_S \psi_n^*(\boldsymbol{r}_1 + \boldsymbol{q})\psi_n(\boldsymbol{r}_2 + \boldsymbol{q})d\boldsymbol{q} = \frac{1}{|\Omega|}J_0\big(k_n|\boldsymbol{r}_1 - \boldsymbol{r}_2|\big), \quad (1.1)$$

provided $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are in the bulk of Ω. Here, J_0 stands for the Bessel function, and $S \subset \Omega$ is a small domain containing many wavelengths k_n^{-1}. The crucial observation is that the same formula is obtained if the genuine eigenfunction ψ_n is replaced with the random function

$$\psi(\boldsymbol{r}, \theta) \equiv \frac{1}{\sqrt{|\Omega|N}}\sum_{j=1}^{N} e^{ik_n^j \boldsymbol{r} + 2\pi i\theta_j}, \quad (1.2)$$

where $\boldsymbol{k}_n^j$ are N wave vectors such that $\|\boldsymbol{k}_n^j\| = k_n$, and θ_j are N independent random variables uniformly distributed on $[0,1]$. In the definition (1.2), the number N of plane waves is considered asymptotically large, and the wave vectors $\boldsymbol{k}_n^j$ cover the circle of radius k_n in an isotropic way. It is not difficult to check that the identity

$$\int \psi^*(\boldsymbol{r}_1,\theta)\psi(\boldsymbol{r}_2,\theta)d\theta = \frac{1}{|\Omega|}J_0\big(k_n|\boldsymbol{r}_1-\boldsymbol{r}_2|\big) \tag{1.3}$$

is indeed satisfied in this large N limit. This equality suggests that all the spatial autocorrelations of an eigenfunction ψ_n might be reproduced by an ensemble average as in (1.3). The ability of such a random waves model to mimic the spatial statistics of the true eigenfunction ψ_n is conjectured in [16]. The typical patterns of the superpositions of random waves (1.2) can be observed in [57]. Notice that each energy e_n requires the introduction of an ensemble of random waves.

The distribution $\mathcal{N}(\psi)$ of the random function ψ in (1.2) is a direct consequence of the central limit theorem. Indeed, the scaling $\sqrt{N}$, which originates from the normalization of ψ, implies that ψ is Gaussian distributed. Hence, the distribution $\mathcal{N}(\psi)$ of ψ is of the form

$$\mathcal{N}(\psi) \propto e^{-\frac{\beta}{2}\int \psi^*(\boldsymbol{r}_1)C^{-1}(\boldsymbol{r}_1,\boldsymbol{r}_2|e_n)\psi(\boldsymbol{r}_2)d\boldsymbol{r}_1 d\boldsymbol{r}_2}, \tag{1.4}$$

where the parameter β is equal to one if time-reversal symmetry is conserved, in which case ψ is real, and is equal to two if this symmetry is broken. Moreover, a Gaussian distribution is characterized by the two-point function it generates, that is, by its covariance. Therefore, by (1.3), Berry's conjecture reads

$$C^{\text{Berry}}(\boldsymbol{r}_1,\boldsymbol{r}_2|e_n) = \frac{1}{|\Omega|}J_0\big(k_n|\boldsymbol{r}_1-\boldsymbol{r}_2|\big). \tag{1.5}$$

This universal Gaussian random waves model is a remarkably good approximation in the bulk of the system, and it finds applications in various domains like optics [19], mesoscopic physics [2], disordered media [7, 56] and quantum chaos [25, 27]. However, the eigenfunctions ψ_n must vanish on the boundary $\partial\Omega$ of

the billiard, and hence, a system-dependent correction must be taken into account near the boundary. A main concern is to know whether this correction can still be incorporated into a Gaussian, yielding this way a non-universal Gaussian model.

In 1998, S. Hortikar and M. Srednicki consider in [48] the function

$$\sum_n \psi_n(\boldsymbol{r}_1)\psi_n^*(\boldsymbol{r}_2)\frac{\delta_\epsilon(e-e_n)}{\bar{\rho}(e)} = -\frac{1}{\pi\bar{\rho}(e)}\Im G(\boldsymbol{r}_1,\boldsymbol{r}_2|e), \tag{1.6}$$

where $\bar{\rho}(e)de$ is the mean number of energy levels in $[e, e+de]$, which is actually independent of e for a two-dimensional billiard, and G is the Green function

$$G(\boldsymbol{r}_1,\boldsymbol{r}_2|e) = \sum_n \frac{\psi_n(\boldsymbol{r}_1)\psi_n^*(\boldsymbol{r}_2)}{e-e_n+i\epsilon}. \tag{1.7}$$

They suggest to replace the correlations (1.5) with the leading order of (1.6) at small $\hbar$. Notice that, contrary to the original random waves model, the modified model obtained in this way is system-dependent and satisfies the Dirichlet boundary conditions on $\partial\Omega$. Moreover, a random function ψ with $\mathcal{N}(\psi) > 0$ still fulfills Helmholtz equation up to ϵ corrections. In the semiclassical approximation, $G(\boldsymbol{r}_1,\boldsymbol{r}_2|e)$ can be expressed as a sum over classical trajectories from $\boldsymbol{r}_2$ to $\boldsymbol{r}_1$ with energy e, as shown in [21] and [45]. This ansatz leads to the sum over classical trajectories

$$C^{HS}(\boldsymbol{r}_1,\boldsymbol{r}_2|e) \propto \frac{1}{\bar{\rho}(e)} \sum_{\substack{p:\ \boldsymbol{r}_2\to\boldsymbol{r}_1 \\ \text{of energy } e}} A_p \cos\left[\frac{S_p}{\hbar} - (2\nu_p+1)\frac{\pi}{4}\right] \tag{1.8}$$

where S_p is the classical action, ν_p is the Maslov index, and A_p is the stability amplitude. It is shown in [48] that Berry's universal result (1.5) is precisely the contribution of the direct path in (1.8). However, the decay of the stability amplitude A_p with the length of the orbits does not compensate the exponentially growing number of these orbits, and hence, infinitely many classical trajectories have to be considered in (1.8).

Four years after [48], M.V. Berry and H. Ishio adapt in [18] and [20] the initial random waves model (1.2) to make it satisfy some fixed boundary conditions along a straight line. They deduce from this new model several statistics

concerning nodal lines and nodal points, and they find unexpected long-ranged boundary effects. Their construction is then generalized in [24] by W.E. Bies, N. Lepore and E.J. Heller to wedge-shaped regions.

In [64, 65, 66], J.D. Urbina and K. Richter reconsider Hortikar and Srednicki's Gaussian system-dependent model. Their main idea is to replace the Dirac function with some energy window w and write the two-point correlation

$$C^{UR}_{e,w}(\boldsymbol{r}_1,\boldsymbol{r}_2) \equiv \sum_n \psi_n(\boldsymbol{r}_1)\psi_n^*(\boldsymbol{r}_2)\frac{w\,(e-e_n)}{\sum_m w\,(e-e_m)}. \tag{1.9}$$

Notice that, within the Gaussian model (1.4) defined from the two-point function (1.9), a random function ψ with $\mathcal{N}(\psi) > 0$ satisfies the Dirichlet boundary conditions on $\partial\Omega$ but does not necessarily fulfill Helmholtz equation. This situation is in this sense opposite to Berry's universal model (1.5). In the semiclassical picture (1.8), the effect of the energy window w in (1.9) is to damp the contributions of the long classical trajectories so that the shortest ones already carry the main contributions. In [66], the numerics show that these short trajectories reproduce the non-universal oscillations of the one-point function near the boundary. Moreover, for large enough energy windows, it is shown in [65] that only the straight classical path contributes, if it exists, and hence, Berry's universal result is obtained. As smaller energy windows are considered, boundary effects become visible, the system-dependent corrections obtained in [18] and [20] are then recovered close to the boundary, and the authors claim that the eigenfunction statistics found in disordered systems by means of the supersymmetry techniques [39] follow from a diagonal approximation of their semiclassical formula.

The implication of Urbina and Richter's waves model is better understood if the random function ψ, which is supposed to mimic an energy eigenfunction ψ_n in the energy window w, is decomposed in the basis $\{\psi_n\}$ of Helmholtz eigenfunctions, namely,

$$\psi(\boldsymbol{r}) \equiv \sum_n a_n\psi_n(\boldsymbol{r}). \tag{1.10}$$

It can be readily checked that the new random variables $\boldsymbol{a} \equiv \{a_n\}$ are independent and distributed according to

$$\mathcal{N}_{\boldsymbol{a}}(\boldsymbol{a}) \propto e^{-\frac{\beta}{2}\sum_n |a_n|^2 C^{UR}_{e,w}(n)^{-1}}, \tag{1.11}$$

where

$$C^{UR}_{e,w}(n) \equiv \frac{w\,(e-e_n)}{\sum_m w\,(e-e_m)}. \tag{1.12}$$

Therefore, according to Urbina and Richter's waves model, the autocorrelations of an energy eigenfunction ψ_n are well reproduced by a random Gaussian superposition of eigenfunctions. In other terms, a single eigenfunction ψ_n looks like a typical linear combination $\sum_m a_m \psi_m$. In particular, for a functional F, one can expect

$$F(\psi_n) \sim \lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^{N} F(\psi_m). \tag{1.13}$$

Such a statement would definitely be completely wrong for the eigenfunctions of an integrable systems.

Quantum graphs are favorable systems to gain some understanding on the possible validity of random waves models. Indeed, on one hand they are in general easier to handle than most systems studied in quantum chaos, and on the other hand, this class of systems contains a broad variety of quantum behaviors, since, as shown in [54, 11, 10, 40, 41], some graphs have energy spectra well described by universal ensembles of random matrices whereas some other graphs do not obey any of these universality classes. Here, without any prior assumption on the nature of the classical dynamics, we can approximate all the moments and all the autocorrelations of the eigenstate intensities, namely,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |\psi_n(\boldsymbol{r}_1)|^2 \cdots |\psi_n(\boldsymbol{r}_p)|^2, \tag{1.14}$$

where $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_p$ are points on the graph. This has to be contrasted with the previous approaches, which all assume the validity of a random waves model. Our results express (1.14) as the sum of an universal component, which obeys some universal Gaussian random waves model, and a system-dependent

correction, which only involves the classical dynamics M on the graph. By comparing these two contributions, we are able to estimate the deviation to universality in terms of M, and to discuss the possible complete failure of the energy eigenfunctions to behave like random waves, as in Neumann star graphs for example [14].

In the case of quantum graphs, the two-point correlation function

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \psi_n^*(\boldsymbol{r}_1)\psi_n(\boldsymbol{r}_2) \tag{1.15}$$

can be exactly calculated. This suffices to completely define the Gaussian random waves model, and since the result happens not to depend on the graph, one deduces that the only Gaussian random waves model that can possibly occur on quantum graphs is universal. Hence, for quantum graphs, the system-dependent component found in (1.14) cannot be incorporated in a Gaussian distribution, and the Gaussian random waves model is bound to exhibit a boundary problem. At first sight, this fact seems to contradict Urbina and Richter's guess. However, their empiric model actually remains compatible with our results concerning quantum graphs since we are dealing with a different kind of system-dependent correction. The systems considered by Urbina and Richter are precisely those expected to be described by some random waves model, namely, the chaotic systems, and their non-universal corrections live close to the boundary. In this work, the system-dependent corrections measure how universal the graph under consideration is, and they can prevail over the universal part, as in Neumann star graphs.

A major achievement would be to show that our result for the autocorrelations (1.14) in the case of quantum graphs also applies to other quantum systems. If this is the case, the deviations from universality vanish in the semiclassical limit in chaotic billiards, which explains why such deviations have indeed never been found, whereas they must prevail over the universal part in non-ergodic systems. For chaotic systems, the corrections would also reveal the approach rate to universality as $\hbar$ asymptotically vanishes. Finally,

if such a formula was found, its ability to describe systems with mixed phase spaces should be studied and compared with the empiric results in [4] and [5].

The moments and autocorrelations of second degree play a particularly important role in quantum chaos, since they suffice to measure the spreading of the energy eigenfunctions, and they can be rigorously controlled. According to A.I. Schnirelman [62], the high energy eigenfunctions of a classically ergodic system should become uniformly spread over the space, a property known as quantum ergodicity. This claim has found rigorous proofs in [31], [30] and [75] for example, where the authors consider compact manifolds with ergodic geodesic flows, quantized ergodic maps and ergodic billiards respectively. The main tool used in these works is an Egorov estimate, which, in the case of quantum maps, reads

$$\left\| U_M^{\dagger k} Op(f) U_M^k - Op(f \circ M^k) \right\| \leq \text{const} \cdot \hbar, \tag{1.16}$$

for M a map, f any smooth function on the configuration space, and U and $Op(f)$ their quantized analogs. A version of (1.16) also holds for continuous Hamiltonian systems. However, the Egorov method does not provide any information on the rate with which quantum ergodicity is reached. This much harder problem is investigated in [71, 73, 74, 33, 3, 61].

In fact, quantum ergodicity is surprisingly more difficult to tackle on quantum graphs than on other chaotic systems. The reason for this difficulty is the non-existence of a deterministic classical map, and hence, of an Egorov estimate. However, in [12], G. Berkolaiko, J.P. Keating and U. Smilansky can prove quantum ergodicity for graphs related to quantum maps by using the Egorov property on the underlying quantum maps. Besides, G. Berkolaiko, J.P. Keating and B. Winn show in [14] that some graphs, namely star graphs, are not quantum ergodic. Here, our result for the autocorrelation functions (1.14) with $p = 2$ enables us to expound a criterion for graphs to become quantum ergodic [42]. Moreover, our method also yields the rate of quantum ergodicity in terms of the classical dynamics, when quantum ergodicity does

occur. The result obtained is a significant refinement of the previous estimates in [33].

The text is structured as follows. The rest of this chapter is devoted to the definition and the basic properties of quantum graphs. In Chapter 2, quantum ergodicity on graphs is defined, and a Gaussian random waves model is built. An exact field theoretic representation of the autocorrelation functions (1.14) is developed in Chapter 3, and in the following two chapters, two contributions are extracted from this exact formula. The sum of these two contributions is then analyzed in Chapter 6, where the main results are obtained, and Chapter 7 concludes and gives possible directions for further research. A great deal of calculations has been put in appendices in order not to interrupt the main guideline provided by the core of the text. Most of the material in these appendices is not the result of original research but aims at summarizing some necessary notions.

1.2 Definitions

In this section and in the next one, the notion of quantum graphs is introduced. In the literature, two different definitions of quantum graphs can be found. In [9], these two approaches are both introduced and compared. The point of view adopted here is the so-called scattering approach. It has been developed for the first time in [54] and is followed in most of the works concerning quantum chaos on graphs. The review [44] offers an extensive introduction to quantum graphs and to their spectral statistics. The second point of view, the Laplace operator approach, can also be found in [29] and in [53], where it was first introduced.

A *metric graph* G is a set of $V \in \mathbb{N}$ points, called *vertices*, and of $B \in \mathbb{N}$ *bonds* of positive lengths $L = (L_1, \cdots, L_B)$ linking some pairs of vertices. The topology of a graph is univoquely determined once its connectivity matrix C

is known. This $V \times V$ matrix reads

$$C_{i,j} = \#\{\text{bonds connecting the vertices } i \text{ and } j\} \tag{1.17}$$

and is symmetric. If $C_{i,i} = 0$ and $C_{i,j} \leq 1$ for all $i, j \in \mathbb{N}_V$, the graph is said *simple*. The *valency* v_i of a vertex $i \in \mathbb{N}_V$ is defined by $v_i = \sum_{j=1}^{V} C_{i,j}$. A point on a graph is specified by a couple (b, x_b), where $b \in \mathbb{N}_B$ determines the bond and $x_b \in [0, L_b]$ determines the position of this point on b. Notice that the parameter x_b naturally defines an orientation on b. However, these orientations only originate from the parametrization of the bonds and they can be arbitrarily chosen. They must not be confused with genuinely orientated bonds, which serve as the building blocks of oriented metric graphs.

A quantum graph is a metric graph G that is turned into a quantum system. In order to do this, the Hilbert space

$$\mathcal{H} = \left\{ \Psi = \bigoplus_{b=1}^{B} \psi_b \middle| \psi_b, \psi_b', \psi_b'' \in L^2([0, L_b]) \right\} \tag{1.18}$$

is introduced, and its elements are referred to as wave functions. Here, the scalar product is

$$(\Psi, \Phi) \equiv \sum_{b=1}^{B} \int_0^{L_b} \psi_b^*(x) \phi_b(x) dx \tag{1.19}$$

for any $\Psi, \Phi \in \mathcal{H}$. The number $\psi_b(x_b)$ is interpreted as the value of the wave function Ψ at $(b, x_b) \in G$. One can define an operator H acting on $\mathcal{H}$ as

$$H\Psi = \bigoplus_{b=1}^{B} (-iD + A_b)^2 \psi_b, \tag{1.20}$$

where D is the derivative operator and $A_b \in \mathbb{R}$. The formula (1.20) is the expression of the Hamiltonian of a particle in a magnetic field in the absence of scalar potential. Notice that the restriction of H on the subset $\mathcal{H}_0 \subset \mathcal{H}$ of wave functions vanishing at the vertices is symmetric. A wave function $\Psi \in \mathcal{H}_0$ is called *Dirichlet wave function.*

For $k > 0$, the elements of $\mathcal{H}$ satisfying $H\Psi = k^2\Psi$ are of the form

$$\Psi \equiv \bigoplus_{b=1}^{B} \psi_b(x_b) = \bigoplus_{b=1}^{B} e^{-iA_b(x_b - L_b/2)} \left(a_{b+} e^{ik(x_b - L_b/2)} + a_{b-} e^{-ik(x_b - L_b/2)} \right). \tag{1.21}$$

Not all these wave functions are physically relevant, and the space $\mathcal{H}$ has thus to be reduced. Firstly, the wave functions are required to be continuous at each vertex $i \in \mathbb{N}_V$. If $\Gamma(i, \text{out})$ and $\Gamma(i, \text{in})$ denote the sets of bonds b having the vertex i at $(b, 0)$ and (b, L_b) respectively, this continuity condition becomes

$$\psi_b(0) = \psi_i \quad \text{and} \quad \psi_{b'}(L_{b'}) = \psi_i \tag{1.22}$$

for some $\psi_i \in \mathbb{C}$, and for all $b \in \Gamma(i, \text{out})$ and $b' \in \Gamma(i, \text{in})$. Secondly, one must ensure the conservation of the probability current

$$j_b(x_b) = \Im\big[\psi_b^*(x_b)\psi_b'(x_b)\big] + A_b|\psi_b(x_b)|^2. \tag{1.23}$$

Any $\Psi \in \mathcal{H}$ conserves this probability current on the interior of every bond, but in order to impose this local conservation at the vertices, namely

$$\sum_{b\in\Gamma(i,\text{out})} j_b(0) + \sum_{b\in\Gamma(i,\text{in})} j_b(L_b) = 0 \tag{1.24}$$

for all $i \in \mathbb{N}_V$, boundary conditions have to be specified. These conditions constrain both the set of possible wavenumbers k and the corresponding amplitudes $\{a_{b\pm}\}$ in (1.21). It is not difficult to check that the wave functions (1.21) satisfying (1.24) are such that

$$\sum_{b\in\Gamma(i,\text{out})} |a_{b+}|^2 = \sum_{b\in\Gamma(i,\text{in})} |a_{b-}|^2, \tag{1.25}$$

Two famous sets of boundary conditions conserving (1.23) are the Dirichlet and the Neumann conditions. The Dirichlet conditions impose to each function ψ_b to vanish at the two extremities $x_b = 0$ and $x_b = L_b$. This completely splits the quantum graph into B independent segments. The Neumann conditions impose continuity and

$$\sum_{b\in\Gamma(i,\text{out})} (iA_b + D)\psi_b(0) + \sum_{b\in\Gamma(i,\text{in})} (iA_b - D)\psi_b(L_b) = 0 \tag{1.26}$$

at each vertex $i \in \mathbb{N}_V$. They are also sometimes referred to as Kirchhoff boundary conditions.

In the sequel, all the magnetic flux A_b are set to zero. The consequence of non-vanishing magnetic flux is discussed in Appendix A.2.

1.3 The Evolution Map

The whole physics of quantum graphs is encapsulated in the wave functions (1.21) which are continuous and conserve the probability current. There is in general no explicit formula for these wave functions and for the corresponding wavenumbers k, but it is still possible to extract valuable information about these quantities. In the scattering approach to quantum graphs, these problems are tackled by means of an evolution matrix $U(k)$.

Let us first consider, for each $k > 0$, the set of $\Psi \in \mathcal{H}$ as in (1.21), that is

$$\tilde{\mathcal{A}}(k) \equiv \left\{ \Psi \in \mathcal{H} \,\middle|\, \Psi = \bigoplus_{b=1}^{B} \left(a_{b+}\tilde{e}_{b+}(k) + a_{b-}\tilde{e}_{b-}(k) \right) \right\}, \tag{1.27}$$

where

$$\tilde{e}_{bd}(k) = e^{idk(x_b - L_b/2)}, \quad \text{for } d \in \{+,-\}. \tag{1.28}$$

On each bond, an element of $\tilde{\mathcal{A}}(k)$ is thus the superposition of two coherent waves traveling in opposite directions. Any wave function $\Psi \in \mathcal{H}$ satisfying $H\Psi = k^2\Psi$ for $k > 0$ belongs to the set $\tilde{\mathcal{A}}(k)$. Let us also introduce $2B$ formal symbols $|e_{bd}\rangle$, where $b \in \mathbb{N}_B$ and $d \in \{+,-\}$, and let us consider the $2B$-dimensional $\mathbb{C}$-linear space,

$$\mathcal{A} = \mathrm{span}\left\{ |e_{bd}\rangle \,\middle|\, b \in \mathbb{N}_B, d \in \{+,-\} \right\}. \tag{1.29}$$

The space $\mathcal{A}$, called *amplitude space*, is endowed with the hermitian scalar product defined by

$$\langle e_{b'd'} | e_{bd} \rangle = \delta_{b,b'}\delta_{d,d'}. \tag{1.30}$$

It can be seen as the product $\mathcal{A}_b \otimes \mathcal{A}_d$ of a B-dimensional *bond space* $\mathcal{A}_b$ and a 2-dimensional *direction space* $\mathcal{A}_d$. There is a natural way to associate an element of $\mathcal{A}$ with each wave function in the set $\tilde{\mathcal{A}}(k)$,

$$\Psi = \bigoplus_{b=1}^{B} \sum_{d=\pm} a_{bd}\tilde{e}_{bd}(k) \quad \mapsto \quad |a\rangle \equiv \sum_{b,d} a_{bd}|e_{bd}\rangle, \tag{1.31}$$

If $\Psi_1 \mapsto |a_1\rangle$ and $\Psi_2 \mapsto |a_2\rangle$, the scalar products of these elements in $\tilde{\mathcal{A}}(k)$ and $\mathcal{A}$ are related by

$$(\Psi_1, \Psi_2) = \left\langle a_1 \left| L + \frac{\sin(kL)}{k} \sigma_1^d \right| a_2 \right\rangle, \tag{1.32}$$

where σ_1^d stands for the first Pauli matrix acting on $\mathcal{A}_d$, and L denotes the $2B \times 2B$ diagonal matrix containing the bond lengths on its diagonal, that is,

$$L_{bd,b'd'} \equiv \delta_{b,b'}\delta_{d,d'}L_b. \tag{1.33}$$

Therefore, length and orthogonality are not preserved by (1.31). Notice that the wave function Ψ can also be recovered from the amplitudes $|a\rangle$ and the wavenumber k, and we thus have a one-to-one map

$$(k, |a\rangle) \mapsto \Psi \equiv \bigoplus_{b=1}^{B} \sum_{d=\pm} a_{bd}\tilde{e}_{bd}(k). \tag{1.34}$$

The construction (1.31) enables to express the conservation of the probability current in a particularly simple way. For any $\Psi \in \tilde{\mathcal{A}}(k)$ and for all vertex $i \in \mathbb{N}_V$, let us form the vectors $|a^i_{\text{out}}\rangle$ from all the numbers $e^{-ikL_b/2}a_{b+}$ with $b \in \Gamma(i, \text{out})$ and $e^{-ikL_b/2}a_{b-}$ with $b \in \Gamma(i, \text{in})$. The vector $|a^i_{\text{out}}\rangle$ defined in this way then contains the values of the waves emerging from vertex i at this vertex. Similarly, we compose the vectors $|a^i_{\text{in}}\rangle$ with all the numbers $e^{ikL_b/2}a_{b-}$ where $b \in \Gamma(i, \text{out})$ and $e^{ikL_b/2}a_{b+}$ where $b \in \Gamma(i, \text{in})$. The vector $|a^i_{\text{in}}\rangle$ then contains the values of the waves incoming to vertex i at this vertex. The scattering process at vertex i can now be described by a $v_i \times v_i$ matrix σ^i,

$$|a^i_{\text{out}}\rangle = \sigma^i |a^i_{\text{in}}\rangle, \tag{1.35}$$

and the condition (1.25) for the conservation of the probability current implies that σ^i must be unitary. In the sequel, we suppose that the V unitary matrices σ^i are given, and that they do not depend on the wavenumber k.

The V vectors $|a^i_{\text{out}}\rangle$ can be grouped together to form a global vector of outgoing values $|a_{\text{out}}\rangle \in \mathbb{C}^{2B}$, and similarly the V vectors $|a^i_{\text{in}}\rangle$ define a global

vector of incoming values $|a_{\text{in}}\rangle \in \mathbb{C}^{2B}$. With these definitions, the V conditions (1.35) can be written in the compact form

$$|a_{\text{out}}\rangle = S|a_{\text{in}}\rangle \quad \text{where} \quad S_{b'd',bd} = \begin{cases} \sigma^i_{b'd',bd} & \text{if } (b,d) \to i \to (b',d') \\ 0 & \text{otherwise} \end{cases} \tag{1.36}$$

Here, the notation $(b,d) \to i \to (b',d')$ means that following the bond b in the direction d leads to the vertex i, and that following b' in the direction $-d'$ reversed to d' also leads to i. The matrix $S \in U(2B)$ is called *scattering matrix* of the quantum graph. It is indeed unitary since all the matrices σ^i are unitary, and it does not depend on the wavenumber k.

Notice that, by definition of the vectors $|a^i_{\text{out}}\rangle$ and $|a^i_{\text{in}}\rangle$, and if $|a\rangle$ denotes the vector obtained from $\Psi \in \tilde{\mathcal{A}}(k)$ by (1.31), then

$$|a_{\text{out}}\rangle = T^\dagger(k)|a\rangle \quad \text{and} \quad |a_{\text{in}}\rangle = T(k)|a\rangle, \tag{1.37}$$

where $T(k) \in U(2B)$ is defined by

$$T(k) \equiv e^{ik\frac{L}{2}} \tag{1.38}$$

and L is as in (1.33). A diagonal component of $T(k)$ contains the phase acquired by a wave traveling along half a bond. This matrix is called *propagation matrix.*

Finally, the equations (1.36) and (1.38) put together imply that

$$U(k)|a\rangle = |a\rangle, \quad \text{where} \quad U(k) \equiv T(k)ST(k), \tag{1.39}$$

must be satisfied. The matrix $U(k) \in U(2B)$ is called *evolution map.* It first propagates the waves along half a bond, then scatters the resulting waves at the vertices, and finally propagates them one more time along half a bond.

Notice that for a non-trivial solution of (1.39) to exist, the *secular equation*

$$\det\Big(1 - U(k)\Big) = 0 \tag{1.40}$$

must be satisfied. This equation first appeared in [54] and is the starting point for studying the properties of quantum graphs. It will be seen after (1.50) that

the wavenumbers satisfying (1.40) form an infinite and diverging sequence of non-negative numbers

$$0 \leq k_0 < k_1 < k_2 < \ldots < k_\nu < k_{\nu+1} < \ldots \to \infty \tag{1.41}$$

called *spectrum* of the quantum graph. It is well-known that the mean number $N(K)$ of elements in the spectrum which lie within the interval $[0, K]$ is given by the *Weyl law*

$$N(K) = K\bar{d} \quad \text{where} \quad \bar{d} \equiv \lim_{K\to\infty} \frac{1}{K} \max\{\nu | k_{\nu-1} < K\} = \frac{\mathrm{tr} L}{2\pi}. \tag{1.42}$$

The parameter $\bar{d}$ is called *mean level density.*

The B bonds of the graph are said to have uncommensurate lengths if

$$\left[n \in \mathbb{Z}^B \quad \text{and} \quad \sum_{b=1}^{B} n_b L_b = 0 \right] \quad \text{implies} \quad n = 0. \tag{1.43}$$

In this case, the spectrum of the quantum graph is generically non-degenerate. In the sequel, only graphs with uncommensurate bond lengths will be considered and, for simplicity, their spectra will be assumed free of degeneracies. It follows that there is a sequence

$$|a^0\rangle,\ |a^1\rangle,\ |a^2\rangle,\ \ldots\ |a^\nu\rangle,\ |a^{\nu+1}\rangle, \ldots \tag{1.44}$$

of normalized vectors in $\mathcal{A}$ such that

$$U(k_\nu)|a^\nu\rangle = |a^\nu\rangle, \tag{1.45}$$

and that each $|a^\nu\rangle$ is unique up to multiplication by $z \in \mathbb{C}$ with $|z| = 1$. Moreover, for $k_\nu > 0$, $|a^\nu\rangle$ precisely contains the $2B$ amplitudes a^ν_{bd} of the wave function $\Psi^\nu \in \mathcal{H}$ satisfying

$$H\Psi^\nu = k_\nu^2 \Psi^\nu. \tag{1.46}$$

For any $k \geq 0$, the evolution map $U(k)$ is unitary, and hence, it exists an orthonormal basis $\{|n, k\rangle | n \in \mathbb{N}_{2B}\}$ of $\mathcal{A}$ and a set of real numbers $\{\phi_n(k) | n \in \mathbb{N}_{2B}\}$ such that

$$U(k)|n, k\rangle = e^{i\phi_n(k)}|n, k\rangle. \tag{1.47}$$

One can order these sets by imposing the inequalities

$$-2\pi < \phi_{2B}(0) \leq \phi_{2B-1}(0) \leq \ldots \leq \phi_2(0) \leq \phi_1(0) \leq 0 \tag{1.48}$$

and by requiring the $2B$ eigencurves $k \mapsto \phi_n(k)$ to be C^∞. This smoothness condition can indeed be realized since the map $U(k)$ depends on k in an analytic way. Taking a derivative with respect to k on both sides of (1.47) easily leads to

$$\phi_n'(k) = \langle n, k|L|n, k\rangle. \tag{1.49}$$

In particular, this shows that

$$0 < L_{\min} \leq \phi_n'(k) \leq L_{\max} < \infty \tag{1.50}$$

where $L_{\min}$ and $L_{\max}$ denote the minimal and maximal bond lengths respectively. Hence, the eigencurves $k \mapsto \phi_n(k)$ are increasing and their slopes are bounded from below and from above. Everytime an eigenphase $\phi_n(k)$ reaches a value $2\pi p$ for some $p \in \mathbb{N}_0$, the quantum map $U(k)$ has an eigenvalue one, and thus k is in the spectrum. Therefore, the spectrum can be indexed by $\nu = (n, p) \in \mathbb{N}_{2B} \times \mathbb{N}_0$, and more importantly, the structure (1.41) of the spectrum follows. Notice that the spectrum is non-degenerate if and only if the eigencurves do not touch each other on $2\pi\mathbb{N}_0$. Numerics show that these curves actually never touch each other and exhibit rather a rigid joint evolution.

Notice that the secular equation (1.40) and the resulting calculus take place in the $2B$-dimensional space of directed bonds $\beta \equiv (b, d) \in \mathcal{A}_b \times \mathcal{A}_d$. One will suppose that an order on the set of directed bonds is defined and write $\beta \in \mathbb{N}_{2B}$ for one element of this set. The privileged basis (1.29) of the amplitude space $\mathcal{A}$ becomes $\{|e_\beta\rangle | \beta \in \mathbb{N}_{2B}\}$. The *origin* $o\beta$ of $\beta \in \mathbb{N}_{2B}$ denotes the vertex from which β emerges, while its *terminus* $t\beta$ is the vertex at which it terminates. Besides, the *reverse* $\hat{\beta}$ of β refers to the directed bond on the same bond as β with opposite direction.

A quantum graph is said *time-reversal invariant* if and only if its evolution map $U(k)$ satisfies

$$U(k)^T \equiv \sigma_1^d U(k)^T \sigma_1^d = U(k), \tag{1.51}$$

where σ_1^d is the first Pauli matrix acting on the direction space $\mathcal{A}_d$, and $U(k)^T$ stands for the transpose of the matrix $U(k)$. Such graphs form the *orthogonal symmetry class*, whereas those which violate (1.51) form the *unitary symmetry class*. An account on time-reversal symmetry in quantum mechanics and especially on quantum graphs is given in Appendix A. If no magnetic field is considered, the orthogonal class is characterized by the condition

$$S^T = S, \quad \text{or } S_{\hat{\beta}'\hat{\beta}} = S_{\beta\beta'} \text{ for all } \beta, \beta' \in \mathbb{N}_{2B} \tag{1.52}$$

on the scattering matrix S.

We finish this section with a remark on the second possible approach to quantum graphs, namely the Laplace operator approach. This method consists in finding all the self-adjoint extensions of H defined on $\mathcal{H}_0$, that is, of the Dirichlet Hamiltonian. It is found that a domain $\mathcal{H}^{\text{sa}}$ making H self-adjoint is composed of the functions $\Psi \in \mathcal{H}$ satisfying some conditions

$$A_i[\Psi(i)] + B_i[\Psi'(i)] = 0 \tag{1.53}$$

at each vertex $i \in \mathbb{N}_V$, where $[\Psi(i)] \in \mathbb{C}^{v_i}$ contains the values of the v_i neighboring functions and $[\Psi'(i)] \in \mathbb{C}^{v_i}$ contains the v_i values of the outgoing derivatives at vertex i. The matrices A_i and B_i lie in $GL(v_i, \mathbb{C})$ and must satisfy some conditions. Firsty, $A_i B_i^\dagger$ must be hermitian, and secondly the $v_i \times 2v_i$ matrix (A_i, B_i) must have maximal rank v_i. All the self-adjoint extensions can be parametrized in this way. Suppose now for simplicity that $\Gamma(i, \text{in}) = \emptyset$. One can then consider the functions $\Psi^i = \bigoplus_{b \in \Gamma(i,\text{out})} \psi_b$ with

$$\begin{aligned} \psi_b(x) &= e^{-ikx} + \sigma^{\text{KS},i}_{b+,b-} e^{ikx} \quad \text{for some } b \in \Gamma(i, \text{out}) \\ \psi_{b'}(x) &= \sigma^{\text{KS},i}_{b'+,b-} e^{ikx} \qquad\quad \text{for } b' \in \Gamma(i, \text{out}) \setminus \{b\} \end{aligned} \tag{1.54}$$

In [53], it is shown that such a function Ψ^i satisfies (1.53) if and only if

$$\sigma^{\text{KS},i} = -(A_i + ikB_i)^{-1}(A_i - ikB_i). \tag{1.55}$$

Therefore, the general self-adjoint boundary conditions (1.53) lead to unitary vertex scattering matrices $\sigma^{\text{KS},i}$ which play the same role as the unitary matri-

ces in (1.35), namely, they describe the wave behavior in the vicinity of the vertices. Then, it is also possible to define a global scattering matrix $S^{\mathrm{KS}} \in U(2B)$ from all the $\sigma^{\mathrm{KS},i}$, and an evolution map $U^{\mathrm{KS}}(k) = T(k)S^{\mathrm{KS}}T(k)$ which characterizes the spectrum by a corresponding secular equation as in (1.40). The major difference between the scattering and the Laplace operator approaches is that, in general, S^{KS} depends on the wavenumber k, as it can be seen on (1.55). However, it is shown in [9] and [29] that any scattering matrix S^{KS} defining a self-adjoint Laplace operator admits a limit S^{KS}_{∞} as k tends to infinity, and moreover, it is argued in [9] that a scattering matrix S^{KS} and its limit S^{KS}_{∞} share the same spectral statistics. The coincidence of these statistics comes from the only fact that they are properties at asymptotically large wavenumber k. Hence, one can deduce that the eigenfunction statistics of S^{KS} and S^{KS}_{∞} also coincide for the same reason.

In the next chapters, the scattering matrix S always refers to the matrix in (1.36) obtained from the scattering approach. It can be any $2B \times 2B$ unitary matrix such that $S_{\beta'\beta}$ vanishes if $t\beta \neq o\beta'$. This class of matrices contains all the asymptotic matrices S^{KS}_{∞} obtained from the Laplace operator approach. There are however some scattering matrices that are acceptable from the scattering point of view but not from the Laplace operator approach. An example is given by the Direct Fourier Transform (DFT) graphs [44], for which the scattering processes at vertex $i \in \mathbb{N}_V$ are described by the $v_i \times v_i$ unitary matrix

$$\sigma^{i}_{\beta'\beta} \equiv \delta_{t\beta,i}\delta_{o\beta',i}e^{2\pi i\frac{n(\beta)n(\beta')}{v_i}}, \tag{1.56}$$

where n assigns to each directed bond $\beta \in \Gamma(i) \equiv \Gamma(i,\mathrm{in}) \cup \Gamma(i,\mathrm{out})$ an integer in $\{0,1,\ldots,v_i-1\}$ such that $n(\hat{\beta}) = n(\beta)$ and $n\big(\Gamma(i)\big) = \{0,1,\ldots,v_i-1\}$. With these boundary conditions, the wavefunctions $\{\Psi^{\nu}\}$ obtained from the amplitudes $\{|a^{\nu}\rangle\}$ and the spectrum $\{k_{\nu}\}$ by (1.34) are not orthogonal to each other in $\mathcal{H}$. This clearly shows that the DFT boundary conditions are not self-adjoint.

A quantum graph is then specified by a pair (G,S) where G is a metric

graph and S is a scattering matrix on G. However, by the remarks made in the preceding two paragraphs, the results concerning the eigenfunction statistics in the Laplace operator approach can be recovered from the formulae in Chapters 4-6 by substituting the k-independent asymptotic matrix S_∞^{KS} for the original k-dependent one S^{KS}.

In the sequel, the metric graphs G considered are assumed simple. The reason for this assumption is to considerably simplify some notations and calculations. However, if a graph contains a directed bond β such that $o\beta = t\beta$, a Neumann vertex can be added on the bond b supporting β to destroy the *loop* b without modifying the quantum dynamics. Similarly, if the graph has two directed bonds β, β' such that $o\beta = o\beta'$ and $t\beta = t\beta'$, a Neumann vertex can be added on the bond b supporting β to destroy this parallel connection without modifying the dynamics. Hence, any graph can be made simple by adding sufficiently many Neumann vertices, and this process does not change the quantum dynamics. One can thus assume the graph simple without loss of generality.

1.4 The Classical Map

In this section, a classical dynamics on graphs is defined, and its main properties are exposed. A more complete treatment can be found in [44].

With any quantum graph, one can associate a *classical map* M defined by

$$M_{\beta\beta'} \equiv |U_{\beta\beta'}(k)|^2 = |S_{\beta\beta'}|^2, \tag{1.57}$$

where $U(k)$ is the evolution map and S is the scattering matrix. The unitarity of S implies that M is bistochastic, that is

$$\sum_\beta M_{\beta\beta'} = \sum_{\beta'} M_{\beta\beta'} = 1. \tag{1.58}$$

This matrix describes a Markov process on the graph, which is the classical counterpart of the quantum dynamics defined by S. The equalities in (1.58)

show that the uniform vector

$$|1\rangle \equiv \frac{1}{\sqrt{2B}} \sum_{\beta=1}^{2B} |e_\beta\rangle \tag{1.59}$$

is an eigenvector of M of eigenvalue 1, and that its hermitian conjugate $\langle 1|$ is a left eigenvectors of M of eigenvalue 1. Besides, Perron-Frobenius theorem [47] ensures that the spectrum of M lies on or within the complex unit disc.

A quantum graph is said to be *dynamically connected* if and only if, for any $\beta, \beta' \in \mathbb{N}_{2B}$, there is $n \in \mathbb{N}$ such that

$$\langle e_{\beta'}|M^n|e_\beta\rangle > 0. \tag{1.60}$$

In other words, a dynamically connected graph is such that there is a possible transition between any two directed bonds β and β' in some discrete time n, which depends on β and β'.

If the graph is dynamically connected, the eigenvalue 1 is non-degenerate, and the graph is *ergodic*, which means that, for any normalized vector $|v\rangle \in \mathcal{A}$ and for any classical observable $f : \mathbb{C}^{2B} \to \mathbb{R}$

$$\frac{1}{N} \sum_{n=0}^{N-1} f\Big(M^n|v\rangle\Big) \overset{N\to\infty}{\longrightarrow} f\Big(|1\rangle\Big). \tag{1.61}$$

Conversely, an ergodic graph is dynamically connected. Any non-ergodic graph (G, S) is the union of several ergodic components, $(G, S) = \bigcup_{i=1}^{k}(G_i, S_i)$. The eigenvalue 1 has degeneracy k, and the k vectors that are uniform on one component and zero on the others form a basis of this eigenspace.

A condition stronger than ergodicity is to be *mixing*, that is

$$M^n|v\rangle \overset{n\to\infty}{\longrightarrow} |1\rangle, \tag{1.62}$$

for any normalized vector $|v\rangle$. The graph is mixing if and only if there is $n \in \mathbb{N}$ such that the inequality (1.60) is satisfied for all directed bonds β and β'. From the spectral point of view, this amounts to require that the eigenvalue 1 is non-degenerate and that no other eigenvalue lies on the unit circle.

Lemma 1.1 *Let M be an ergodic $2B \times 2B$ stochastic matrix, and let $M = D_M + N_M$ be its Jordan decomposition into a diagonalizable part D_M and a nilpotent part N_M commuting with each other. Let $\lambda_j \in \mathbb{C}$, $j \in \mathbb{N}_{2B}$, be the $2B$ eigenvalues of D_M, and let $|j\rangle \in \mathbb{C}^{2B}$ be corresponding normalized eigenvectors. Then, $\langle 1|j\rangle = \delta_{1,j}$, where $|1\rangle$ is given by (1.59).*

Proof. The matrix M being stochastic, $\langle 1|$ is a left eigenvector of M of eigenvalue 1, and by ergodicity, this eigenvalue is non-degenerate. Moreover, since M, D_M and N_M commute with each other, and since N_M is nilpotent, $\langle 1|$ is also a left eigenvector of D_M of eigenvalue 1. Therefore,

$$\langle 1|j\rangle(1-\lambda_j) = \langle 1|D_M - \lambda_j|j\rangle = 0. \tag{1.63}$$

□

This lemma shows that the classical map M can be viewed as the sum of an uniform contribution $|1\rangle\langle 1|$ and a remainder R', which satisfies $\langle 1|R' = 0$ and $R'|1\rangle = 0$.

This decomposition can be brought to the level of sums of classical orbits on the graph. Let $M_\epsilon = e^{-2\epsilon}M$ for $\epsilon > 0$. The sum of all classical orbits from β to β' followed with M_ϵ is nothing else but

$$\left(\frac{M_\epsilon}{1-M_\epsilon}\right)_{\beta',\beta} = \left(M_\epsilon + M_\epsilon^2 + M_\epsilon^3 + \ldots\right)_{\beta',\beta}. \tag{1.64}$$

These expressions turn out to be essential to tackle some properties of classical graphs such as return probabilities. They are precisely the object of the next corollary.

Corollary 1.2 *Let M denote an ergodic classical map, and let $M_\epsilon = e^{-2\epsilon}M$ for some $\epsilon > 0$. Then*

$$\frac{M_\epsilon}{1-M_\epsilon} = \frac{e^{-2\epsilon}}{1-e^{-2\epsilon}}|1\rangle\langle 1| + R_\epsilon,$$

where R_ϵ is a $2B \times 2B$ real matrix such that the massive component *$R \equiv \lim_{\epsilon\to 0} R_\epsilon$ exists and satisfies $\langle 1|R = 0$ and $R|1\rangle = 0$. In particular,*

$$trR = \sum_{i=2}^{2B} \frac{1-m_i}{m_i},$$

where the masses $m_i \equiv 1 - \lambda_i$, $i \in \mathbb{N}_{2B}$, *are the eigenvalues of* $\mathbb{1} - D_M$, D_M *being the diagonalizable part of* M *as in Lemma 1.1. The masses* m_i *all lie in the closed disc of radius 1 and centered at 1 in the complex plane, and the zero mass* $m_1 = 0$ *is non-degenerate.*

Proof. By the preceding lemma, it exists a change of orthonormal bases $\{|e_\beta\rangle\} \to \mathcal{B}$ described by a unitary matrix U such that

$$U^{-1}(1 - M_\epsilon)U = \left(\begin{array}{c|ccc} 1 - e^{-2\epsilon} & \cdots & 0 & \cdots \\ \hline \vdots & & & \\ 0 & & \star_\epsilon & \\ \vdots & & & \end{array}\right), \quad \star_\epsilon \in \mathrm{Mat}(2B-1, \mathbb{R}). \quad (1.65)$$

The first vector of $\mathcal{B}$ is $|1\rangle$, so that the endomorphism defined from M_ϵ by replacing the matrix $\star_\epsilon$ with the zero matrix is nothing else but $(1-e^{-2\epsilon})|1\rangle\langle 1|$. Besides, the eigenvalues of $\star_\epsilon$ do not vanish. Let us write (1.65) in the form

$$1 - M_\epsilon = (1 - e^{-2\epsilon}) \oplus \star_\epsilon, \quad (1.66)$$

where the number before the direct sum refers to the coefficient in front of $|1\rangle\langle 1|$ whereas the endomorphism after the sum is the restriction of $1 - M_\epsilon$ onto the subspace generated by $|2\rangle, \ldots, |2B\rangle$. For an analytic function $f : D \subset \mathbb{R} \to \mathbb{R}$, the matrix valued function $f(1 - M_\epsilon)$ is defined by the Taylor series, and it has the same block diagonal form as (1.65) once written in the basis $\mathcal{B}$. Thus, one has the decomposition

$$f(1 - M_\epsilon) = f(1 - e^{-2\epsilon}) \oplus f(\star_\epsilon). \quad (1.67)$$

In particular

$$\frac{M_\epsilon}{1 - M_\epsilon} = \frac{e^{-2\epsilon}}{1 - e^{-2\epsilon}} \oplus \frac{1 - \star_\epsilon}{\star_\epsilon} = \frac{e^{-2\epsilon}}{1 - e^{-2\epsilon}}|1\rangle\langle 1| + R_\epsilon, \quad (1.68)$$

where we have defined $R_\epsilon = 0 \oplus \frac{1-\star_\epsilon}{\star_\epsilon}$. Besides, it is straightforward to check that $R \equiv \lim_{\epsilon \to 0} R_\epsilon$ fulfills all the properties enounced in the corollary.

□

Notice that if a quantum graph (G, S) is time-reversal invariant, that is if S fulfills (1.52), then its classical map satisfies $M = M^T \equiv \sigma_1^d M^T \sigma_1^d$.

Chapter 2

Statistics of the Eigenfunctions

2.1 Quantum Ergodicity

2.1.1 The Wave Definition

The definition of quantum ergodicity requires the prior introduction of some notations. Let G be a metric graph with B bonds. An *observable on* G is a family

$$V = \left\{ V_b \in C^0([0, L_b]) \middle| b \in \mathbb{N}_B \right\} \tag{2.1}$$

of B real functions $V_b(x)$ defined on the bonds of G. The mean value $\bar{V}$ of an observable V is defined by

$$\bar{V} \equiv \frac{2}{\mathrm{tr}L} \int_G^{\oplus} V \equiv \frac{2}{\mathrm{tr}L} \sum_{b=1}^{B} \int_0^{L_b} V_b(x) dx. \tag{2.2}$$

Notice that $\frac{\mathrm{tr}L}{2} = \int_G^{\oplus} 1$ is the volume of G. In the sequel, the observables that are constant on each bond will be particularly important. If an observable V is constant on each bond, one can merely write $V = (V_b)_{b \in \mathbb{N}_B}$ with $V_b \in \mathbb{R}$. The mean value of such an observable reads

$$\bar{V} = \frac{\sum_{b=1}^{B} V_b L_b}{\sum_{b=1}^{B} L_b} \tag{2.3}$$

and is invariant under a global scaling of the bond lengths.

Suppose now that $S \in U(2B)$ is a scattering matrix on G. The quantum graph (G, S) is *quantum ergodic* if and only if it exists a subsequence $i \mapsto \nu(i)$ of density 1 such that

$$\lim_{i\to\infty} \frac{\left(\Psi^{\nu(i)}, V\Psi^{\nu(i)}\right)}{\left(\Psi^{\nu(i)}, \Psi^{\nu(i)}\right)} = \bar{V} \tag{2.4}$$

for any observable V. In this definition, $\Psi^\nu = \bigoplus_{b=1}^B \psi_b^\nu$ denotes an eigenfunction of H of eigenvalue k_ν^2.

The left-hand side of (2.4) represents the mean value of the observable V in the eigenstate $\Psi^{\nu(i)}$. A straightforward calculation shows that

$$\begin{aligned}(\Psi^\nu, V\Psi^\nu) &= \sum_{b=1}^B \left(|a_{b+}^\nu|^2 + |a_{b-}^\nu|^2\right) \int_0^{L_b} V_b(x)dx \\ &\quad + 2\Re \sum_{b=1}^B a_{b-}^{\nu *} a_{b+}^\nu \int_0^{L_b} V_b(x) e^{2ik_\nu\left(x-\frac{L_b}{2}\right)} dx \end{aligned} \tag{2.5}$$

for the wavefunction Ψ^ν with wavenumber $k_\nu > 0$ and amplitudes a_{b+}^ν and a_{b-}^ν as in (1.34). Since the observable V is assumed continuous on each bond, and since $|a_{b-}^{\nu *} a_{b+}^\nu| \leq 1$, the second term in the right-hand side of (2.5) is of order $\mathcal{O}(k_\nu^{-1})$ as k_ν becomes asymptotically large. Hence, this second term does not affect the left-hand side of (2.4) where the high energy limit is taken. Moreover, the first term in the right-hand side of (2.5) remains unchanged if the observable V is replaced with the observable W defined by $W_b \equiv L_b^{-1} \int_0^{L_b} V_b(x)dx$. These two remarks imply that, in the definition (2.4) of quantum ergodicity, it is actually sufficient to only consider observables that are constant on each bond, and this will always be done in the sequel.

Moreover, if the equality (2.4) holds for any observable of vanishing mean $\bar{V} = 0$, then it also holds for any observable W. In order to see this, it is sufficient to observe that $W - \bar{W}$ has vanishing mean and to apply (2.4) to this new observable. Hence, without loss of generality, one can restrict our attention to observables V with $\bar{V} = 0$.

If the identity (2.4) is satisfied for any subsequence of eigenfunctions, the quantum graph is said *quantum unique ergodic*. In [60], it is shown that many

short closed cycles, like the triangle $\beta_1 \to \beta_2 \to \beta_3 \to \beta_1$ for instance, support eigenfunctions with arbitrarily high energies. These eigenfunctions, called *scars*, break quantum unique ergodicity. Hence, this stronger property cannot be expected on quantum graphs in general, and only quantum ergodicity will be investigated in the sequel. However, it is possible to build graphs where the shortest periodic orbit without back-scattering has period of order $\log V$ [28], V denoting the number of vertices. For such graphs, the scarring phenomenon weakens as V becomes large, and the quantum unique ergodicity issue could also be investigated.

It will be seen in Chapter 4 that quantum ergodicity cannot be expected to be realized on a finite quantum graph. This notion has thus to be replaced with a weaker one called asymptotic quantum ergodicity. Let us consider an infinite sequence $\{(G_l, S_l)\}_{l\in\mathbb{N}}$ of quantum graphs with increasing number of bonds $B_l < B_{l+1}$. We also suppose that the bonds of any G_l have lengths in $[L_{\min}, L_{\max}]$, where $0 < L_{\min} < L_{\max} < \infty$ are independent of l. Such a sequence will be called *increasing*. We always assume that either all the graphs (G_l, S_l) are time-reversal invariant, or they all break this symmetry. The eigenfunctions of (G_l, S_l) are denoted by Ψ_l^ν, and similarly, all the quantities introduced in Chapter 1 are indexed by l. Besides, a sequence $\{V_l\}_{l\in\mathbb{N}}$, where V_l is an observable on G_l, is said *acceptable* if and only if the two conditions

$$\begin{gathered} \lim_{l\to\infty} \bar{V}_l \equiv \bar{V}_\infty \text{ exists,} \\ 0 \leq |V_{l,b}| \leq V_{\max} \end{gathered} \tag{2.6}$$

are fulfilled. Then, an increasing sequence $\{(G_l, S_l)\}_{l\in\mathbb{N}}$ of quantum graphs is *asymptotically quantum ergodic* if and only if

$$\lim_{l\to\infty} \lim_{i\to\infty} \frac{\left(\Psi_l^{\nu(i)}, V_l \Psi_l^{\nu(i)}\right)}{\left(\Psi_l^{\nu(i)}, \Psi_l^{\nu(i)}\right)} = \bar{V}_\infty \tag{2.7}$$

for all acceptable sequence of observables $\{V_l\}_{l\in\mathbb{N}}$. The limit $l \to \infty$ plays the role of semiclassical limit for quantum graphs.

For the sequences of graphs satisfying (2.7), the rate of convergence is also

of particular interest. Therefore, we will work with a single finite quantum graph at a time, and come back to convergence and rate considerations at the very end.

2.1.2 The Amplitude Definition

A calculation similar to (2.5) shows that, for an observable V on G constant on each bond, one has

$$\begin{aligned}(\Psi^\nu, V\Psi^\nu) &= \left\langle a^\nu \left| VL\left(1+\frac{\sin(k_\nu L)}{k_\nu L}\right)\right| a^\nu\right\rangle \\ &= \langle a^\nu|VL|a^\nu\rangle + \mathcal{O}(k_\nu^{-1}),\end{aligned} \tag{2.8}$$

where $|a^\nu\rangle \in \mathcal{A}$ is the vector of amplitudes defining Ψ^ν through the construction (1.34). In order to get the second equality, $|a^\nu_{b+}a^\nu_{b-}| \leq 1$ has been used. Hence, the expectation value of an observable V in the eigenstate Ψ^ν becomes

$$\frac{(\Psi^\nu, V\Psi^\nu)}{(\Psi^\nu, \Psi^\nu)} = \frac{\langle a^\nu|VL|a^\nu\rangle}{\langle a^\nu|L|a^\nu\rangle} + \mathcal{O}(k_\nu^{-1}). \tag{2.9}$$

There is in (2.8) and (2.9) a slight abuse of notation. In the left-hand side of (2.9), V is an observable constant on each bond, whereas in the right-hand side V is the square matrix of size $2B$ such that $V_{bd,b'd'} = \delta_{b,b'}\delta_{d,d'}V_b$, where V_b is the value of the observable V on bond b.

The definition (2.1) of observable on a metric graph G and the equality (2.9) induce what an observable must be in the amplitude framework. An *observable on* $\mathcal{A}$ is a real diagonal matrix $V = \mathrm{diag}(V_\beta)_{\beta\in\mathbb{N}_{2B}}$ which is independent of the direction, that is $V_{\hat\beta} = V_\beta \equiv V_b$, where b is the bond supporting β and $\hat\beta$. The mean value $\bar V$ of V is

$$\bar V \equiv \frac{\mathrm{tr}(VL)}{\mathrm{tr}L} = \frac{\sum_{\beta=1}^{2B} V_\beta L_\beta}{\sum_{\beta=1}^{2B} L_\beta} = \frac{\sum_{b=1}^{B} V_b L_b}{\sum_{b=1}^{B} L_b}. \tag{2.10}$$

There is an obvious correspondence between observables on G that are constant on each bond and observables on $\mathcal{A}$. The construction is precisely the one given at the end of the last paragraph. The formulae (2.3) and (2.10) show that the

mean value of V does not depend on whether V is seen as an observable on G or on $\mathcal{A}$.

The relation (2.9) between expectation values in the wave and in the amplitude contexts enables to express the quantum ergodicity condition (2.4) in the amplitude space $\mathcal{A}$. Indeed, a quantum graph (G, S) is quantum ergodic if and only if it exists a subsequence $i \mapsto \nu(i)$ of density 1 such that

$$\lim_{i\to\infty} \frac{\langle VL\rangle_{\nu(i)}}{\langle L\rangle_{\nu(i)}} \equiv \lim_{i\to\infty} \frac{\langle a^{\nu(i)}|VL|a^{\nu(i)}\rangle}{\langle a^{\nu(i)}|L|a^{\nu(i)}\rangle} = \bar{V} \tag{2.11}$$

for any observable V on $\mathcal{A}$.

The two reasons for which the equivalence between the wave and the amplitude definitions (2.4) and (2.11) of quantum ergodicity holds are, firstly, because quantum ergodicity is a property at high energy, and secondly, because the discrepancy in the expectation values (2.9) vanishes at high energy.

In the amplitude formulation, the acceptable sequences $\{V_l\}_{l\in\mathbb{N}}$ of observables are those satisfying the two conditions in (2.6). Asymptotic quantum ergodicity for an increasing sequence $\{(G_l, S_l)\}_{l\in\mathbb{N}}$ of quantum graphs becomes

$$\lim_{l\to\infty}\lim_{i\to\infty} \frac{\langle V_l L_l\rangle_{l,\nu(i)}}{\langle L_l\rangle_{l,\nu(i)}} \equiv \lim_{l\to\infty}\lim_{i\to\infty} \frac{\langle a_l^{\nu(i)}|V_l L_l|a_l^{\nu(i)}\rangle}{\langle a_l^{\nu(i)}|L_l|a_l^{\nu(i)}\rangle} = \bar{V} \tag{2.12}$$

for a subsequence $i \mapsto \nu(i)$ and for all acceptable $\{V_l\}_{l\in\mathbb{N}}$.

In (2.11) and (2.12), one can always restrict our attention to $\bar{V} = 0$ without loss of generality, in the same manner as with the corresponding wave formulations.

Moreover, it is clear that quantum graphs that are not ergodic cannot be quantum ergodic. Indeed, suppose that a graph (G, S) is the union of two components (G_1, S_1) and (G_2, S_2) with $B_1 > 0$ and $B_2 > 0$ bonds respectively. Then, the spectrum of (G, S) is the union of the spectra of (G_1, S_1) and (G_2, S_2), and one of these components, say (G_1, S_1), supports a subsequence of energy eigenfunctions $i \mapsto \nu(i)$ of positive density. For some $V_0 > 0$, let us choose $V_\beta = B_2 V_0$ for β in G_1 and $V_\beta = -B_1 V_0$ for β in G_2. Then, the mean value of V in any eigenstate $|a^{\nu(i)}\rangle$ is $B_2 V_0$, which does not reach $\bar{V} = 0$ in the limit $i \to \infty$. Hence, only ergodic graphs will be considered in the sequel.

2.1.3 Local Weyl Law and Fluctuations

On the way to tackle asymptotic quantum ergodicity on graphs, one is confronted to two difficulties. The first obstacle is that, in general, no explicit formula is known for the amplitudes $|a^\nu\rangle$ of the eigenfunctions. When such a formula does exist, as in [14] for example, a direct analysis then leads to the conclusion. The second apparent difficulty is that one has no information on the location of the scars in the spectrum. In other terms, there is no obvious candidate for the convenient subsequence of density 1 of eigenfunctions. A general method to overcome the second difficulty, used in [75] and [3] for example, is presented in this subsection, and the Green matrix formalism developed in Section 2.3 takes care of the first obstacle.

One can extract from the quantum ergodicity property (2.11) a weaker statement, known as *local Weyl law*, which reads

$$A_V \equiv \lim_{K\to\infty} \frac{1}{N(K)} \sum_{k_\nu \leq K} \frac{\langle VL\rangle_\nu}{\langle L\rangle_\nu} = \bar{V} \tag{2.13}$$

for all observables V. In (2.13), the high energy limit considered in the definition (2.11) of quantum ergodicity is replaced with an average over all the eigenfunctions. If quantum ergodicity is fulfilled, then this average merely takes the value of the high energy limit, and the local Weyl law is then satisfied. However, the sequence

$$x_\nu \equiv \frac{\langle VL\rangle_\nu}{\langle L\rangle_\nu} - \bar{V} \tag{2.14}$$

can oscillate around zero so that $A_V = \bar{V}$ and not converge to zero. In order to tackle the quantum ergodicity issue, (2.13) has thus to be complemented with a statement about the fluctuations of the sequence $x = \{x_\nu\}_{\nu\in\mathbb{N}}$ for any observable V on $\mathcal{A}$. For any real sequence $y = \{y_\nu\}_{\nu\in\mathbb{N}}$, and for $m \in \mathbb{N}$, one introduces the real number

$$M_m(y) \equiv \frac{1}{m}\sum_{\nu=1}^{m} y_\nu. \tag{2.15}$$

Then, if $M(y)$ denotes the limit of $M_m(y)$ as $m \to \infty$ when this limit exists, the local Weyl law reads $M(x) = 0$, where $x = \{x_\nu\}_{\nu\in\mathbb{N}}$ is given by (2.14). Notice also that the sequence x is bounded. Indeed, for all $\nu \in \mathbb{N}$, one has $|x_\nu| \leq L_{\min}^{-1} V_{\max} L_{\max} + \bar{V}$. The following lemma, proved in [67], enables to relate quantum ergodicity to the fluctuations of x.

Lemma 2.1 *Let $x = \{x_\nu\}_{\nu\in\mathbb{N}}$ be a bounded real sequence. Then the two following assertions are equivalent.*

- *It exists a subsequence $i \mapsto \nu(i)$ of density 1 such that $x_{\nu(i)} \to 0$ as $i \to \infty$.*
- *$M(x^2) = 0$, where $x^2 \equiv \{x_\nu^2\}_{\nu\in\mathbb{N}}$.*

This result can be applied to the sequence $x = \{x_\nu\}_{\nu\in\mathbb{N}}$ defined in (2.14). It implies that quantum ergodicity, as defined in (2.11), is fulfilled if and only if the fluctuations

$$F_V \equiv \lim_{K\to\infty} \frac{1}{N(K)} \sum_{k_\nu \leq K} \left(\frac{\langle VL \rangle_\nu}{\langle L \rangle_\nu} - \bar{V} \right)^2 \tag{2.16}$$

vanish for all observables V on $\mathcal{A}$. Notice that the fluctuations F_V do not refer to the possible problematic sequence of scarred eigenfunctions anymore. Similarly, an increasing sequence of quantum graphs $\{(G_l, S_l)\}_{l\in\mathbb{N}}$ is asymptotically quantum ergodic if and only if the fluctuations

$$F_{l,V_l} \equiv \lim_{K\to\infty} \frac{1}{N_l(K)} \sum_{k_{l,\nu} \leq K} \left(\frac{\langle V_l L_l \rangle_{l,\nu}}{\langle L_l \rangle_{l,\nu}} - \bar{V}_l \right)^2 \tag{2.17}$$

decay to zero as $l \to \infty$ for all acceptable sequences of observables $\{V_l\}_{l\in\mathbb{N}}$.

In particular, the fluctuations of an observable V of vanishing mean on a single graph (G, S) read

$$F_V = \lim_{K\to\infty} \frac{1}{N(K)} \sum_{k_\nu \leq K} \frac{\langle VL \rangle_\nu^2}{\langle L \rangle_\nu^2} \tag{2.18}$$

It will be seen in Section 2.3 that the quantity

$$\mathcal{F}_V = \lim_{K\to\infty} \frac{1}{N(K)} \sum_{k_\nu \leq K} \frac{2B}{\mathrm{tr} L \langle L \rangle_\nu} \langle VL \rangle_\nu^2 \tag{2.19}$$

can be expressed in terms of Green functions, and is thus easier to analyze than the original fluctuations (2.18). Besides, since by assumption the bond lengths lie between $L_{\min}$ and $L_{\max}$,

$$\frac{L_{\min}}{L_{\max}} F_V \leq \mathcal{F}_V \leq \frac{L_{\max}}{L_{\min}} F_V, \tag{2.20}$$

and therefore, $F_V = 0$ if and only if $\mathcal{F}_V = 0$. One deduces that an increasing sequence $\{(G_l, S_l)\}_{l\in\mathbb{N}}$ of quantum graphs is asymptotically quantum ergodic if and only if

$$\lim_{l\to\infty} \mathcal{F}_{l,V_l} = 0 \tag{2.21}$$

for all acceptable $\{V_l\}_{l\in\mathbb{N}}$. Moreover, in the case of asymptotic quantum ergodicity, F_{l,V_l} and $\mathcal{F}_{l,V_l}$ have the same convergence rate.

In Section 2.2, several statistical quantities are introduced. In order to make the notations used for quantum ergodicity compatible with this more general setup, the fluctuations (2.19) are written

$$\mathcal{F}_V = \left(\frac{2B}{\mathrm{tr}L}\right)^2 \sum_{\beta,\beta'=1}^{2B} (VL)_\beta (VL)_{\beta'} \left\langle |a_\beta|^2 |a_{\beta'}|^2 \right\rangle . \tag{2.22}$$

By comparison with (2.19), (2.22) implicitely defines

$$\left\langle |a_\beta|^2 |a_{\beta'}|^2 \right\rangle \equiv \lim_{K\to\infty} \frac{1}{N(K)} \sum_{k_\nu \leq K} \frac{\mathrm{tr}L}{2B\langle L\rangle_\nu} |a_\beta^\nu|^2 |a_{\beta'}^\nu|^2. \tag{2.23}$$

The surprising factor $\frac{\mathrm{tr}L}{2B\langle L\rangle_\nu}$ in (2.23) can be given a precise meaning. Indeed, the next theorem claims that performing an average over the spectrum of the quantum graph in presence of this factor, such as in (2.23) for example, amounts to average the same quantity over all the eigenfunctions $|n,k\rangle$ of $U(k)$ and then to integrate over all $k \in (0,\infty)$. The reader can refer to [15] for the proof. In the following, the notation $\langle O\rangle_{n,k} = \langle n,k|O|n,k\rangle$ is used.

Theorem 2.2 *Suppose that the bond lengths are uncommensurate. Then, for any observable O on $\mathcal{A}$, and any $m > 0$,*

$$\lim_{K\to\infty} \frac{1}{N(K)} \sum_{k_\nu \leq K} \frac{trL}{2B\langle L\rangle_\nu} \langle O\rangle_\nu^m = \lim_{K\to\infty} \frac{1}{K} \int_0^K \frac{1}{2B} \sum_{n=1}^{2B} \langle O\rangle_{n,k}^m dk.$$

From the preceding theorem with $m = 1$, the average A_V in (2.13) becomes

$$\begin{aligned} A_V &= \lim_{K\to\infty} \frac{1}{N(K)} \sum_{k_\nu \le K} \frac{\mathrm{tr}L}{2B\langle L\rangle_\nu} \left\langle \frac{2B}{\mathrm{tr}L} VL \right\rangle_\nu \\ &= \lim_{K\to\infty} \frac{1}{K} \int_0^K \frac{1}{2B} \sum_{n=1}^{2B} \left\langle \frac{2B}{\mathrm{tr}L} VL \right\rangle_{n,k} \\ &= \frac{\mathrm{tr}VL}{\mathrm{tr}L} \equiv \bar{V}. \end{aligned} \tag{2.24}$$

Therefore, the local Weyl law (2.13) holds for any finite graph. Two alternative proofs of this result are given in 2.1.4 and in 2.3.2.

2.1.4 Trace Formulae

The question of quantum ergodicity can be addressed by means of trace formulae, that is, by means of periodic orbits on the graph. In this subsection, an alternative proof of the local Weyl law is given, and then, the fluctuations $\mathcal{F}_V$ in (2.19) are approximated.

Let $\epsilon > 0$, and let us introduce the matrix-valued function on $\mathbb{R}_+$

$$\mathcal{G}_\epsilon(k) \equiv \sum_{n=1}^{2B} |n,k\rangle\langle n,k| \cdot \delta_{\epsilon,2\pi}\big(\phi_n(k)\big), \tag{2.25}$$

where $\delta_{\epsilon,2\pi}$ stands for

$$\delta_{\epsilon,2\pi}(x) \equiv \sum_{p\in\mathbb{Z}} \delta_\epsilon(x - 2\pi p) \quad \text{with} \quad \delta_\epsilon(x) \equiv \frac{\epsilon}{\pi}\frac{1}{x^2+\epsilon^2}, \quad x \in \mathbb{R}. \tag{2.26}$$

Both the average A_V in (2.13) and the fluctuations $\mathcal{F}_V$ in (2.19) can be written in terms of $\mathcal{G}_\epsilon(k)$. Indeed, one has

$$A_V = \lim_{\epsilon\to 0} \frac{2\pi}{\mathrm{tr}L} \Big\langle \mathrm{tr}\big(\mathcal{G}_\epsilon(k)VL\big) \Big\rangle_k, \tag{2.27}$$

and

$$\mathcal{F}_V = \lim_{\epsilon\to 0} \frac{(2\pi)^2 2B\epsilon}{(\mathrm{tr}L)^2} \Big\langle \big[\mathrm{tr}\big(\mathcal{G}_\epsilon(k)VL\big)\big]^2 \Big\rangle_k. \tag{2.28}$$

where the *spectral average* $\langle \ldots \rangle_k$ is defined by

$$\langle \ldots \rangle_k \equiv \lim_{K\to\infty} \frac{1}{K} \int_0^K \ldots \, dk. \tag{2.29}$$

We prove here the identity (2.28), the previous one being easier to obtain. One starts with the observation

$$\frac{2\pi\epsilon}{K}\int_0^K \left[\mathrm{tr}\big(\mathcal{G}_\epsilon(k)VL\big)\right]^2 = \frac{2\pi\epsilon}{K}\int_0^K \sum_{n,m=0}^{2B}\sum_{p,q=0}^{\infty}\langle VL\rangle_{n,k}\langle VL\rangle_{m,k} \cdot \delta_\epsilon(\phi_n(k)-2\pi p)\delta_\epsilon(\phi_m(k)-2\pi q). \quad (2.30)$$

Then, the fact that the distribution $2\pi\epsilon\cdot\delta_\epsilon(x)\delta_\epsilon(y)$ tends to zero as $\epsilon\to 0$ if $x\neq y$ and to $\delta_\epsilon(x)$ if $x=y$ can be used. This identity between distributions shows that only the terms with $n=m$ and $p=q$ contribute to the right-hand side of (2.30). Notice that this conclusion relies on the non-degeneracy of the spectrum. Similar ideas are commonly used to tackle the distribution of matrix elements [70]. Now, if the equation (2.30) is divided by $\bar{d}$ given in (1.42), and if the limit $\epsilon\to 0$ is taken, one obtains

$$\begin{aligned}\frac{(2\pi)^2\epsilon}{K\mathrm{tr}L}\int_0^K \mathrm{tr}\big(\mathcal{G}_\epsilon(k)VL\big)^2 &\xrightarrow{\epsilon\to 0} \frac{1}{N(K)}\int_0^K \sum_{n=1}^{2B}\langle VL\rangle_{n,k}^2\,\delta_{2\pi}(\phi_n(k)) \\ &= \frac{1}{N(K)}\int_0^K \sum_{n=1}^{2B}\langle VL\rangle_{n,k}^2\sum_{p=0}^{\infty}\frac{\delta(k-k_{n,p})}{\phi_n'(k_{n,p})} \\ &= \frac{1}{N(K)}\sum_{k_\nu\le K}\frac{\langle VL\rangle_\nu^2}{\langle L\rangle_\nu}. \end{aligned} \quad (2.31)$$

Here, the notation $k_{n,p}$ refers to the characterization of the spectrum exposed after (1.50), and the last equality comes from the identity (1.49). Finally, multiplying (2.31) by the factor $\frac{2B}{\mathrm{tr}L}$, and taking the limit $K\to\infty$ terminates the proof of (2.28).

The main step to obtain trace formulae for A_V and $\mathcal{F}_V$ is to apply the Poisson summation formula to the matrix $\mathcal{G}_\epsilon(k)$. This formula gives

$$\sum_{p\in\mathbb{Z}}\delta_\epsilon(\phi_n(k)-2\pi p) = \sum_{q\in\mathbb{Z}}\frac{e^{i\phi_n(k)q}}{2\pi}e^{-\epsilon|q|} \quad (2.32)$$

This result is now plugged into the definition (2.25) of $\mathcal{G}_\epsilon$, and the mean part

$q = 0$ is separated from the oscillatory part $q \neq 0$. This yields

$$\begin{aligned} \mathcal{G}_\epsilon(k) &= \frac{1}{2\pi}\sum_{n=1}^{2B}|n,k\rangle\langle n,k| + \frac{1}{2\pi}\sum_{n=1}^{2B}|n,k\rangle\langle n,k|\sum_{q=1}^{\infty}\left(e^{i\phi_n(k)q}+e^{-i\phi_n(k)q}\right)e^{-\epsilon q} \\ &= \frac{1}{2\pi}\mathbb{1} + \frac{1}{\pi}\Re\sum_{q=1}^{\infty}U(k)^q e^{-\epsilon q}. \end{aligned} \tag{2.33}$$

Second Proof of the Local Weyl Law

In this paragraph, a proof of (2.13) using the representation (2.33) is presented. Let us first introduce some terminology. An *oriented path* $\vec{\beta}$ is a list $(\beta_0, \beta_1, \ldots, \beta_n)$ of consecutive directed bonds on the graph. The *topological length* of such a $\vec{\beta}$ is $|\vec{\beta}| = n$, which corresponds to the number of vertices traversed by $\vec{\beta}$. The set of all oriented paths having topological length n is written C_n. The subset of C_n consisting of closed paths is denoted by C_n^0, and $\cup_{n\in\mathbb{N}}C_n^0 \equiv C^0$. The *metric length* of $\vec{\beta}$ is

$$l(\vec{\beta}) \equiv \frac{L_{\beta_0}}{2} + \sum_{i=1}^{n-1}L_{\beta_i} + \frac{L_{\beta_n}}{2}. \tag{2.34}$$

Only half the lengths of β_0 and β_n are included in the definition of $l(\vec{\beta})$ since the oriented path $\vec{\beta}$ is imagined to start at the middle point of β_0 and to terminate at the middle point of β_n. The origin and terminus of $\vec{\beta}$ are respectively $o\vec{\beta} \equiv \beta_0$ and $t\vec{\beta} \equiv \beta_n$. We also define the *stability amplitude*

$$A_{\vec{\beta}} \equiv \prod_{i=0}^{n-1} S_{\beta_{i+1},\beta_i}. \tag{2.35}$$

By (2.33), one can write

$$\begin{aligned} \mathrm{tr}\,(\mathcal{G}_\epsilon VL) &= \frac{\mathrm{tr}(VL)}{2\pi} + \frac{1}{\pi}\Re\sum_{q=1}^{\infty}e^{-\epsilon q}\mathrm{tr}\,(U^qVL) \\ &= \frac{\mathrm{tr}(VL)}{2\pi} + \frac{1}{\pi}\Re\sum_{q=1}^{\infty}e^{-\epsilon q}\sum_{\substack{\beta_i=(b_i,d_i)\\ i=0\ldots q-1}}(VL)_{\beta_0}U_{\beta_0,\beta_{q-1}}\ldots U_{\beta_1,\beta_0} \\ &= \frac{\mathrm{tr}(VL)}{2\pi} + \frac{1}{\pi}\Re\sum_{q=1}^{\infty}e^{-\epsilon q}\sum_{\vec{\beta}\in C_q^0}V_{o\vec{\beta}}e^{ikl(\vec{\beta})}A_{\vec{\beta}}. \end{aligned} \tag{2.36}$$

Since all the path lengths $l(\vec{\beta})$ are positive for $|\vec{\beta}| \geq 1$, and since $\langle e^{ikl} \rangle_k = 0$ if $l \neq 0$, only the first term in the last expression survives the spectral average. Then, multiplying by the suitable factor shows that A_V in (2.27) is given by

$$A_V = \frac{2\pi}{\mathrm{tr}L} \lim_{\epsilon \to 0} \Big\langle \mathrm{tr}\big(\mathcal{G}_\epsilon(k) V L\big) \Big\rangle_k = \frac{\mathrm{tr}(VL)}{\mathrm{tr}L} \tag{2.37}$$

which is precisely the local Weyl law.

Long Diagonal Orbits

The fluctuations $\mathcal{F}_V$ can be expressed in terms of closed paths by means of the formula (2.33). In this paragraph, one will only retain a subset of the whole collection of closed paths that contribute to this expression. The strategy here is strongly inspired from [33]. The resulting formula predicts the decay B^{-1} for $\mathcal{F}_V$ as $B \to \infty$.

We suppose that $\bar{V} = 0$, so that the first term in (2.36) vanishes. The second term of this formula can be written in terms of periodic orbits p rather than in terms of closed paths $\vec{\beta}$. A periodic orbit can be seen as an equivalence class of closed paths whose sequences of directed bonds differ from each other by cyclic permutations. In the sequel, the value O_p of an observable O on the periodic orbit p denotes the number obtained by cumulating the values O_β of O along p. The formula (2.36) is then represented by the sum over periodic orbits

$$\mathrm{tr}\,(\mathcal{G}_\epsilon V L) = \frac{1}{\pi} \Re \sum_p e^{-\epsilon|p|} \frac{(VL)_p}{r_p} e^{ikl_p} A_p, \tag{2.38}$$

Here, the repetition number r_p is the number of times p retraces itself, and the notations $|p|$, l_p and A_p are inherited from the closed path terminology exposed in the previous paragraph.

The square of (2.38) admits the spectral average

$$\Big\langle \big[\mathrm{tr}(\mathcal{G}_\epsilon V L)\big]^2 \Big\rangle_k = \frac{1}{2\pi^2} \sum_{p,q:l_p=l_q} \frac{(VL)_p (VL)_q}{r_p r_q} \Re(A_p A_q^*) e^{-\epsilon(|p|+|q|)}. \tag{2.39}$$

In the terms $p \neq q$, the real parts fluctuate considerably around zero. By contrast, in the terms $p = q$, these real parts are always positive, and hence, these

terms bring a systematic contribution. Moreover, if time-reversal invariance is conserved, the stability amplitudes of an orbit p and of its reversed $\hat{p}$ coincide. Therefore, in this case, the pairs $(p, \hat{p})$ also contribute. The diagonal approximation consists in only keeping the pairs $q = p$, and $q = \hat{p}$ in the time-reversal invariant case. It yields

$$\left\langle \left[\mathrm{tr}(\mathcal{G}_\epsilon VL)\right]^2 \right\rangle_k^{\mathrm{diag}} = \frac{\kappa}{2\pi^2} \sum_{p,q:l_p=l_q} \frac{\left[(VL)_p\right]^2}{r_p^2} |A_p|^2 e^{-2\epsilon|p|}, \tag{2.40}$$

where $\kappa = 1$ if time-reversal invariance is broken and $\kappa = 2$ if this symmetry is conserved.

The formula (2.40) is then approximated further. The orbits for which $r_p > 1$ are rare, and we will thus only keep the primitive orbits, namely those with $r_p = 1$. We also take the long orbits approximation [33]. Suppose that a periodic orbit p is represented by the closed oriented path $\vec{\beta} \equiv (\beta_0, \beta_1, \ldots, \beta_n)$, $\beta_n = \beta_0$, and for $k \in \mathbb{Z}$, write $(VL)_p(k) \equiv (VL)_{\beta_{j(k)}}$, where $j(k) \in \{0, \ldots, n-1\}$ is such that $j(k) \equiv k \pmod{n}$. With this notation, one has

$$\left[(VL)_p\right]^2 = \sum_{k,l=0}^{n-1} (VL)_p(k)(VL)_p(l) = \sum_{c,\Delta} (VL)_p(c+\Delta)(VL)_p(c-\Delta), \tag{2.41}$$

where we set $k = c + \Delta$ and $l = c - \Delta$. On the right-hand side, the sums over c and Δ have to be constrained such that each pair $\left(j(c+\Delta), j(c-\Delta)\right)$ is represented exactly once. The last expression can also be written as a sum over possible point c

$$\left[(VL)_p\right]^2 = \sum_c C_p(c) \tag{2.42}$$

of *correlations* between values of $(VL)_p$ at points with midpoint c, that is

$$C_p(c) \equiv \sum_\Delta (VL)_p(c+\Delta)(VL)_p(c-\Delta). \tag{2.43}$$

For orbits that are long compared to the size $2B$ of the graph, one can typically expect that all the possible pairs of values of VL enter $C_p(c)$ as Δ runs from 1 to its upper limit. In this case, only the term $\Delta = 0$ brings a systematic contribution since, by assumption, $\mathrm{tr}VL = 0$. Hence, the long orbits approximation

leads to $C_p(c) \sim (VL)_p(c)^2$, which, together with (2.41), implies that

$$\left[(VL)_p\right]^2 \sim \left[(VL)^2\right]_p \sim |p| \frac{\mathrm{tr}(VL)^2}{2B}. \tag{2.44}$$

The last relation is also part of the long orbits approximation. It amounts to replace the value of $(VL)^2$ along the orbit by the mean value of $(VL)^2$ multiplied by the length of the orbit. Besides, the stability amplitude is known to behave like [44]

$$|A_p|^2 \sim e^{-\alpha|p|}, \tag{2.45}$$

where α is the topological entropy. This parameter also characterizes the number $|p|^{-1}e^{\alpha|p|}$ of periodic orbits of topological length $|p|$. With all these approximations, (2.40) generates the integral

$$\left\langle \left[\mathrm{tr}(\mathcal{G}_\epsilon VL)\right]^2 \right\rangle_k^{\mathrm{diag}} \sim \frac{\kappa}{2\pi^2} \int_0^\infty \frac{e^{\alpha|p|}d|p|}{|p|} \cdot |p| \frac{\mathrm{tr}(VL)^2}{2B} \cdot e^{-\alpha|p|} \cdot e^{-2\epsilon|p|}, \tag{2.46}$$

and multiplying both sides of this relation by the suitable number, the formula (2.28) generates the expression

$$\mathcal{F}_V^{\mathrm{diag}} \sim \kappa \frac{\mathrm{tr}(VL)^2}{(\mathrm{tr}L)^2}. \tag{2.47}$$

The argument presented here tends to show that the fluctuations vanish like B^{-1} as $B \to \infty$, since V and L have bounded components and the denominator in (2.47) involves one more trace factor than the numerator. Moreover, the formula (2.47) also predicts that the approach to quantum ergodicity is twice faster if time-reversal invariance is conserved than if it is broken. In fact, (2.47) is a part of the result that will be obtained with the more elaborated supersymmetry method. An additional system-dependent part of this result will leave the possibility for the fluctuations not to decay. This possible non-universal behavior is due to short primitive orbits, which have not been taken into account in this argument.

2.2 Random Waves Models

2.2.1 Unitary Symmetry

Let us consider $2B$ complex random variables a_β, and let us investigate the existence of a joint probability density $\varphi(\boldsymbol{a}) = \varphi(a_1, \ldots, a_{2B})$ satisfying

$$\left\langle \prod_{k=1}^{p} a^*_{\beta_k} \prod_{l=1}^{q} a_{\beta'_l} \right\rangle \equiv \lim_{K\to\infty} \frac{1}{N(K)} \sum_{k_\nu \le K} \frac{\mathrm{tr}L}{2B\langle L\rangle_\nu} \prod_{k=1}^{p} a^{\nu *}_{\beta_k} \prod_{l=1}^{q} a^{\nu}_{\beta'_l} \quad (2.48)$$

$$= \int_{\mathbb{C}^{2B}} \prod_{k=1}^{p} a^*_{\beta_k} \prod_{l=1}^{q} a_{\beta'_l}\, \varphi(\boldsymbol{a})\, d\boldsymbol{a}^* d\boldsymbol{a}, \quad (2.49)$$

for any choice of $\beta_1, \ldots, \beta_p, \beta'_1, \ldots, \beta'_q \in \mathbb{N}_{2B}$. Here, the measure $d\boldsymbol{a}^* d\boldsymbol{a}$ denotes the product of the $2B$ flat Lebesgue measures $da^*_\beta da_\beta$ in the complex plane. Notice that $\langle 1\rangle = 1$, so that $\varphi(\boldsymbol{a})$, if it exists, is indeed a probability density. In order to see that, one can make use of Theorem 2.2 and write

$$\lim_{K\to\infty} \frac{1}{N(K)} \sum_{k_\nu \le K} \frac{\mathrm{tr}L}{2B\langle L\rangle_\nu} = \lim_{K\to\infty} \frac{1}{K} \int_0^K \frac{1}{2B} \sum_{n=1}^{2B} 1 = 1 \quad (2.50)$$

Besides, the same theorem shows that the mean values defined in (2.48) with the factor $\frac{\mathrm{tr}L}{2B\langle L\rangle_\nu}$ correspond to averages over the set of eigenfunctions $|n,k\rangle$ of $U(k)$ with a subsequent integral over $k \in (0,\infty)$. A main concern is to know whether $\varphi(\boldsymbol{a})$ could possibly coincide with the normal probability density

$$\mathcal{N}_\sigma(\boldsymbol{a}) d\boldsymbol{a}^* d\boldsymbol{a} \equiv \prod_{\beta=1}^{2B} \mathcal{N}_{\sigma,\beta}(a_\beta) da^*_\beta da_\beta \equiv \prod_{\beta=1}^{2B} \frac{1}{4\pi\sigma^2_\beta}\, e^{-\frac{|a_\beta|^2}{2\sigma^2_\beta}}\, da^*_\beta da_\beta. \quad (2.51)$$

for some well-chosen $\sigma = (\sigma_1, \ldots, \sigma_{2B})$. If $\varphi(\boldsymbol{a})$ exists and is equal to $\mathcal{N}_\sigma(\boldsymbol{a})$, the density $\mathcal{N}_\sigma(\boldsymbol{a})$ provides us with a Gaussian random waves model that enables one to describe the statistical properties of the eigenfunctions.

It is often more convenient to consider the polar random variables $|a_\beta|^2 \in (0,\infty)$ and $\theta_\beta \in [0, 2\pi)$ instead of $a^*_\beta = |a_\beta| e^{-i\theta_\beta}$ and $a_\beta = |a_\beta| e^{i\theta_\beta}$. The corresponding Jacobian being one, one has

$$\mathcal{N}_{\sigma,\beta}(a_\beta) da^*_\beta da_\beta = e^{-\frac{|a_\beta|^2}{2\sigma^2_\beta}}\, d\left(\frac{|a_\beta|^2}{2\sigma^2_\beta}\right) \cdot \frac{d\theta_\beta}{2\pi}. \quad (2.52)$$

In fact, it can be directly realized that such a Gaussian model cannot be exact for finite B. Indeed, $\sum_\beta |a_\beta|^2$ is a random variable with positive variance whereas $|a^\nu\rangle$ lies on the unit sphere S^{4B-1} in $\mathbb{C}^{2B}$, which implies that the complex random variables a_β cannot be totally independent. Similarly, the modulus of an amplitude $|a^\nu_\beta|$ cannot exceed one, whereas $\mathcal{N}_{\sigma,\beta}(a_\beta)$ is positive even for $|a_\beta| > 1$. However, it is not impossible that, for $1 \ll B$ and $p+q \ll B$, $\langle a^*_{\beta_1} \dots a^*_{\beta_p} a_{\beta'_1} \dots a_{\beta'_q} \rangle$ is well approximated by a Gaussian model, and that this approximation becomes exact in the limit $B \to \infty$.

Establishing the validity of a Gaussian model using the method of moments would require the calculation of (2.48) for arbitrary products of amplitudes. We concentrate here on the products of the kind

$$C_\beta(q) \equiv \langle |a_{\beta_1}|^{2q_1} \dots |a_{\beta_n}|^{2q_n} \rangle \tag{2.53}$$

where $n \in \mathbb{N}$, $q = (q_1, \dots, q_n)$ is a vector containing n positive integers and $\beta = (\beta_1, \dots, \beta_n)$ is a list of n directed bonds. The quantities $C_\beta(q)$ will subsequently be referred to as *autocorrelation functions of the amplitudes*. They completely determine the statistical behavior of the moduli $|a_\beta|^2$. An autocorrelation of the type (2.48) that does not only depend on moduli $|a_\beta|^2$ or on products of the type $a^*_\beta a_{\hat{\beta}}$ can be expected to vanish since each term k_ν contains additional phases that strongly fluctuate and take part in the average. This expectation is indeed realized in any Gaussian model of the type (2.51). In particular, this shows that one can restrict our attention to autocorrelation functions (2.48) with $p = q$.

In particular, the *moments* read

$$M_\beta(q) \equiv C_\beta(q) = \langle |a_\beta|^{2q} \rangle . \tag{2.54}$$

The *degree* of the autocorrelation function $C_\beta(q)$ is

$$q \equiv \sum_{k=1}^{n} q_k \tag{2.55}$$

and corresponds to the number of factors $|a_\beta|^2$ in (2.53).

Another notation for the autocorrelations $C_{\beta}(\boldsymbol{q})$ turns out to be more appropriate for the subsequent calculations. One associates with the vectors $\boldsymbol{\beta}$ and $\boldsymbol{q}$ as above the extended list $[\boldsymbol{\alpha}] \equiv [\alpha_0, \ldots, \alpha_{q-1}]$ of q directed bonds defined by

$$\alpha_s = \begin{cases} \beta_1 & \text{if } 0 \leq s \leq q_1 - 1 \\ \beta_2 & \text{if } q_1 \leq s \leq q_1 + q_2 - 1 \\ \vdots & \vdots \\ \beta_n & \text{if } \sum_{k=1}^{n-1} q_k \leq s \leq q - 1 \end{cases} \tag{2.56}$$

and one writes

$$C_{[\boldsymbol{\alpha}]} \equiv C_{\boldsymbol{\beta}}(\boldsymbol{q}) \equiv \left\langle |a_{\beta_1}|^{2q_1} \ldots |a_{\beta_n}|^{2q_n} \right\rangle = \left\langle |a_{\alpha_0}|^2 \ldots |a_{\alpha_{q-1}}|^2 \right\rangle . \tag{2.57}$$

There are some statistical quantities which are not of the type $C_{\boldsymbol{\beta}}(\boldsymbol{q})$ and which can be very easily obtained. Indeed, a slight variation of the calculation (2.24) based on Theorem 2.2 implies

$$\langle a_{\beta}^* a_{\beta'} \rangle = \frac{\delta_{\beta,\beta'}}{2B}. \tag{2.58}$$

Notice that for $\beta \neq \beta'$ this result is in agreement with any Gaussian model of the type (2.51). Besides, (2.58) shows that the mean value of $|a_{\beta}|^2$ reads

$$M_{\beta}(1) = \frac{1}{2B} \tag{2.59}$$

for all $\beta \in \mathbb{N}_{2B}$. This implies that the only random waves model of the type (2.51) that could possibly be satisfied in the limit of large graphs is the one with variances

$$2\sigma_{\beta}^2 = \frac{1}{2B}. \tag{2.60}$$

Therefore, the only Gaussian model of the type (2.51) that really has to be considered is

$$\mathcal{N}(\boldsymbol{a}) d\boldsymbol{a}^* d\boldsymbol{a} \equiv \prod_{\beta=1}^{2B} e^{-2B|a_{\beta}|^2} d\left(2B|a_{\beta}|^2\right) \cdot \frac{d\theta_{\beta}}{2\pi}. \tag{2.61}$$

In this formula, all the directed bonds are treated in the same way so that this Gaussian model is not system-dependent. The moments and the first

autocorrelations of this function read

$$\langle |a_\beta|^{2q} \rangle_{\mathcal{N}} = \frac{q!}{(2B)^q} \quad \text{and} \quad \langle |a_\beta|^2 |a_{\beta'}|^2 \rangle_{\mathcal{N}} = \frac{1}{(2B)^2} \tag{2.62}$$

for any directed bonds $\beta \neq \beta'$ in $\mathbb{N}_{2B}$. Here and in the sequel, the purpose of the subscript $\mathcal{N}$ is to distinguish between the statistical quantities coming from the Gaussian model $\mathcal{N}(a)$ in (2.61) and the quantities coming from an actual calculation or approximation of (2.48). The predictions of the Gaussian model concerning more general autocorrelation functions $C_\beta(q)$ are accessible from the moments in (2.62) invoking the independence of the random variables $|a_\beta|^2$ and $|a_{\beta'}|^2$ for $\beta \neq \beta'$. From these two formulae, it is direct to check that the joint probability density function (2.61) has the enviable property

$$\left\langle \sum_{\beta=1}^{2B} |a_\beta|^2 \right\rangle_{\mathcal{N}} = 1. \tag{2.63}$$

However, as mentioned above, $\sum_\beta |a_\beta|^2$ is a random variable with positive variance. Indeed, the two identities in (2.62) yield

$$\left\langle \left(\sum_{\beta=1}^{2B} |a_\beta|^2 - 1 \right)^2 \right\rangle_{\mathcal{N}} = \frac{1}{2B}. \tag{2.64}$$

Notice that this expression vanishes in the limit $B \to \infty$, leaving the possibility for the Gaussian model (2.61) to be asymptotically satisfied in the limit of large graphs.

In fact, the identity (2.58) also shows that a system-dependent Gaussian model cannot be realized. Indeed, suppose that $\mathcal{N}_\sigma$ in (2.51) is replaced with

$$\mathcal{N}'_\sigma(a) da^* da \propto e^{-\sum_{\beta,\beta'} \frac{a^*_\beta}{\sqrt{2}\sigma_\beta} \left[\delta_{\beta,\beta'} + C_{\beta\beta'} \right] \frac{a_{\beta'}}{\sqrt{2}\sigma_{\beta'}}}. \tag{2.65}$$

Then,

$$\langle a^*_\beta a'_\beta \rangle_{\mathcal{N}'} \propto 2\sigma_\beta \sigma_{\beta'} \left[\delta_{\beta,\beta'} + C_{\beta\beta'} \right], \tag{2.66}$$

and a comparison with (2.58) shows that $C_{\beta\beta'} = 0$ for all $\beta \neq \beta'$. Hence, C must be diagonal and the preceding arguments show that the only possibility corresponds to $2\sigma_\beta^2 = (2B)^{-1}$ and $C = 0$.

2.2.2 Orthogonal Symmetry

In the previous section, time-reversal invariance has not been mentioned, and all the questions addressed there are sensible for both unitary and orthogonal graphs. However, if time-reversal invariance is conserved, the random variables a^*_β and $a_{\hat{\beta}}$ should not be independent from each other, contradicting the initial guess (2.51). In this subsection, an universal Gaussian waves model is built for the orthogonal symmetry class starting from the unitary Gaussian model (2.61) and two intuitive expectations.

The first expectation is that the eigenvector statistics of a graph in the orthogonal class should not discern an amplitude a_β from the amplitude $a_{\hat{\beta}}$ supported on the reverse directed bond. This claim is indeed verified in 2.3.2. In particular, the equality

$$\langle |a_\beta|^4 \rangle = \langle |a_{\hat{\beta}}|^2 |a_\beta|^2 \rangle. \tag{2.67}$$

is fulfilled for any directed bond $\beta \in \mathbb{N}_{2B}$. The second natural expectation is that an autocorrelation function $\langle |a_{b_1 d_1}|^2 \dots |a_{b_q d_q}|^2 \rangle$ does not depend on the symmetry class if the bonds $b_1, \dots, b_q$ are all different from each other. Notice that (2.58) establishes this assertion in the case $q = 1$.

The two preceding expectations tend to show that, in the orthogonal case, the eigenfunction statistics is better expressed in terms of the B random variables

$$i_b \equiv |a_{b+}|^2 + |a_{b-}|^2, \tag{2.68}$$

representing the B *intensities* on the non-directed graph. Therefore, in the orthogonal case, the goal is to investigate the existence of a joint probability density function $\phi(\boldsymbol{i})$ such that

$$\left\langle \prod_{k=1}^{n} i_{b_k}^{q_k} \right\rangle = \int_{[0,\infty)^B} \prod_{k=1}^{n} i_{b_k}^{q_k} \, \phi(\boldsymbol{i}) \, d\boldsymbol{i} \tag{2.69}$$

for any set of bonds $\{b_1, \dots, b_n\}$ and any $q_1, \dots, q_n \in \mathbb{N}$. The average in the left-hand side of (2.69) is an average over the spectrum of the graph as defined in (2.48).

The natural candidate for the joint probability density function $\phi(i)$ is thus

$$\phi(i) = \prod_{b=1}^{B} e^{-Bi_b}\, d(Bi_b). \tag{2.70}$$

The moments and the first autocorrelation functions of $\phi(i)$ read

$$\langle i_b^q \rangle_\phi = \frac{q!}{B^q} \quad \text{and} \quad \langle i_b i_{b'} \rangle_\phi = \frac{1}{B^2} \tag{2.71}$$

for any bonds $b \neq b'$. More general statistical quantities as in (2.69) can be obtained from the moments in (2.71) by independency of the variables i_b and $i_{b'}$ for $b \neq b'$.

The predictions (2.71) of the Gaussian model (2.70) for orthogonal symmetry have exactly the same form as the corresponding predictions (2.62) of the Gaussian model (2.61) for unitary symmetry. In summary, the suitable random waves model giving simple statistics as in (2.62) and (2.71) concerns directed bonds in the unitary case, whereas it concerns non-directed bonds in the orthogonal case.

In fact, $\phi(i)$ in (2.70) cannot be obtained as the result of an underlying joint probability density function of the amplitudes a_{b+} and a_{b-}. Indeed, in order to get the wanted properties (2.67), and the mean values $\langle |a_\beta|^2 \rangle$ as in the unitary case, this joint probability density function should be of the form

$$\prod_{b=1}^{B} \left(\frac{B}{\pi}\right)^2 e^{-2B\left(\begin{smallmatrix} a_{b+}^* & a_{b-}^* \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a_{b+} \\ a_{b-} \end{smallmatrix}\right)} da_{b+}^* da_{b+} da_{b-}^* da_{b-}. \tag{2.72}$$

Each factor of this product contains the two complex amplitudes living on some bond b and should thus be integrated over $\mathbb{C}^2$. Let us consider one of these factors, and let us introduce the real parts $r_\pm = \Re(a_{b\pm})$ and the imaginary parts $j_\pm = \Im(a_{b\pm})$ in view of integration. The factor b of (2.72) then becomes

$$\left(\frac{2B}{\pi}\right)^2 e^{-2Bv^T M v} dr_+ dr_- dj_+ dj_-, \tag{2.73}$$

where

$$v \equiv \begin{pmatrix} r_+ \\ r_- \\ j_+ \\ j_- \end{pmatrix}, \quad M \equiv \begin{pmatrix} 1 & 1 & 0 & i \\ 1 & 1 & i & 0 \\ 0 & -i & 1 & 1 \\ -i & 0 & 1 & 1 \end{pmatrix}. \tag{2.74}$$

The differential (2.73) has now to be integrated over $\mathbb{R}^4$. But this integral diverges since the determinant of the hermitian matrix M is negative. This shows that (2.72) is not a joint probability density function, and hence, the random waves model in the orthogonal symmetry case cannot be written in terms of amplitudes.

2.2.3 Predictions Concerning Ergodicity

The unitary and orthogonal Gaussian models presented in the previous two subsections both predict formulae for the fluctuations $\mathcal{F}_V$ of an observable V. Suppose that the unitary Gaussian model is satisfied, then

$$\langle |a_\beta|^2 |a_{\beta'}^2| \rangle = \frac{1 + \delta_{\beta,\beta'}}{(2B)^2}. \tag{2.75}$$

This leads to

$$\begin{aligned} \mathcal{F}_V &= \left(\frac{1}{\mathrm{tr}L}\right)^2 \left[\sum_{\beta \neq \beta'} (VL)_\beta (VL)_{\beta'} + 2\sum_\beta (VL)_\beta^2\right] \\ &= \left(\frac{1}{\mathrm{tr}L}\right)^2 \left[\sum_{\beta,\beta'} (VL)_\beta (VL)_{\beta'} + \sum_\beta (VL)_\beta^2\right] \\ &= \left(\frac{\mathrm{tr}(VL)}{\mathrm{tr}L}\right)^2 + \frac{\mathrm{tr}(VL)^2}{(\mathrm{tr}L)^2}. \end{aligned} \tag{2.76}$$

In particular, if $\bar{V} = 0$, only the second term survives. This result shows that the unitary Gaussian model predicts that any increasing sequence of graphs becomes asymptotically quantum ergodic as $B \longrightarrow \infty$. Moreover, it agrees with the prediction (2.47) of the long diagonal orbits.

Suppose now that the orthogonal Gaussian model (2.70) is met. Then, the intensities satisfy

$$\langle i_b i_{b'} \rangle = \frac{1 + \delta_{b,b'}}{B^2}. \tag{2.77}$$

Therefore,

$$\begin{aligned}
\mathcal{F}_V &= \left(\frac{2B}{\mathrm{tr}L}\right)^2 \sum_{b,b'} (VL)_b (VL)_{b'} \langle i_b i_{b'} \rangle_\phi \\
&= \left(\frac{2}{\mathrm{tr}L}\right)^2 \left[\sum_{b \neq b'} (VL)_b (VL)_{b'} + 2\sum_b (VL)_b^2\right] \\
&= \left(\frac{2}{\mathrm{tr}L}\right)^2 \left[\sum_{b,b'} (VL)_b (VL)_{b'} + \sum_b (VL)_b^2\right] \\
&= \left(\frac{\mathrm{tr}(VL)}{\mathrm{tr}L}\right)^2 + 2 \cdot \frac{\mathrm{tr}(VL)^2}{(\mathrm{tr}L)^2}. \qquad (2.78)
\end{aligned}$$

Hence, if the obervable VL is traceless, the fluctuations $\mathcal{F}_V$ are twice larger in the orthogonal case than in the unitary case, which also agrees with the prediction (2.47).

2.2.4 Obstructions to the Universal Gaussian Models

The universal random waves models (2.61) and (2.70) cannot be exactly realized on a graph. Indeed, as already mentioned, the norm of the random variable $a \in \mathbb{C}^{2B}$ has positive variance, whereas the amplitudes $|a^\nu) \in \mathbb{C}^{2B}$ of any eigenvector lie on the unit sphere S^{4B-1}. Moreover, there is also a discrepancy between the random variables $|a_\beta|^2$ and the corresponding amplitudes $|a_\beta^\nu|^2$ since the first are not restricted to lie in the unit disc. However, as the size of the graph increases, the probability for $|a_\beta|^2$ to be larger than any fixed $\delta > 0$ decreases exponentially. Hence, it is possible that, for any given β and q, the autocorrelation function $C_\beta(q)$ tends to the Gaussian predictions in the limit of large graphs.

There is another obstruction to the random waves models (2.61) and (2.70). The boundary condition at vertex i impose some correlation between the amplitudes supported on the neighboring bonds. This type of local and system-dependent correlations is ignored in the universal Gaussian models (2.61) and (2.70). The most striking example consists in adding a Neumann vertex on some bond b of an ergodic graph. By doing so, the bond b is split into two new

bonds b_1 and b_2, which can be oriented such that $(b_1, +) \to (b_2, +)$. Then, the Neumann boundary condition imposes $|a_{b_1+}|^2 = |a_{b_2+}|^2$ and $|a_{b_1-}|^2 = |a_{b_2-}|^2$. These strong correlations contradict the predictions (2.62) and (2.71). Hence, a necessary condition for the universal Gaussian models (2.61) and (2.70) to be fulfilled in the limit of large graph is that all the valencies tend to infinity. This situation is similar to the chaotic billiards where the universal Gaussian waves model discussed in Section 1.1 provides accurate predictions while violating the Dirichlet boundary conditions. In this comparison with billiards, the vertices have a double role. They are similar to the boundary of the billiard, but, at the same time, their required infinite valencies play the role of the infinite possible directions for the wave vectors, which lead, by the central limit theorem, to the Gaussian models.

2.3 Green Matrices

2.3.1 Product of Green Matrices

The Green matrices defined below and their products constitute the main tool that will enable us to tackle the eigenfunction statistics introduced in (2.48) in the forthcoming chapters.

First, for $\epsilon > 0$, the sub-unitary evolution map $U_\epsilon(k) \equiv e^{-\epsilon}U(k)$ is introduced. It is sometimes convenient to let the scattering matrix S carry the ϵ factor, in which case one writes

$$U_\epsilon(k) = T(k)S_\epsilon T(k) \quad \text{with} \quad S_\epsilon \equiv e^{-\epsilon}S. \tag{2.79}$$

The *retarded Green matrix* $G(k)$ is the matrix-valued function on $\mathbb{R}_+$ defined by

$$G(k) = \Big(1 - U_\epsilon(k)\Big)^{-1} = \sum_{n=1}^{2B} \frac{|n,k\rangle\langle n,k|}{1 - e^{i(\phi_n(k)+i\epsilon)}}. \tag{2.80}$$

Notice that this matrix has poles at $\phi_n(k) = 2\pi p - i\epsilon$, $p \in \mathbb{Z}$. The *advanced*

Green matrix $G^\dagger(k)$ is the hermitian conjugate of $G(k)$, that is

$$G^\dagger(k) = \left(1 - U_\epsilon^\dagger(k)\right)^{-1} = \sum_{n=1}^{2B} \frac{|n,k\rangle\langle n,k|}{1 - e^{-i(\phi_n(k)-i\epsilon)}}. \tag{2.81}$$

It has poles at $\phi_n(k) = 2\pi p + i\epsilon$, $p \in \mathbb{Z}$. The goal is to make use of these matrices to express the quantity (2.48) with $p = q$, which can be rewritten

$$\begin{aligned}
&\langle a^*_{\beta_1} \dots a^*_{\beta_q} a_{\beta'_1} \dots a_{\beta'_q} \rangle \\
&= \lim_{K\to\infty} \frac{1}{Kd} \int_0^K \sum_{n=1}^{2B} \sum_{p=0}^{\infty} \frac{\delta(k - k_{n,p}) \mathrm{tr} L}{2B\langle n,k|L|n,k\rangle} \prod_{j=1}^{q} \langle e_{\beta_j}|n,k\rangle\langle n,k|e_{\beta'_j}\rangle \\
&= \frac{\pi}{B} \lim_{K\to\infty} \frac{1}{K} \int_0^K \sum_{n=1}^{2B} \delta_{2\pi}(\phi_n(k)) \prod_{j=1}^{q} \langle e_{\beta_j}|n,k\rangle\langle n,k|e_{\beta'_j}\rangle
\end{aligned} \tag{2.82}$$

There are actually several ways to achieve this goal, but a particularly convenient way is given by the next theorem. Similar methods have also been used to study eigenfunction statistics in disordered systems [38].

Theorem 2.3 *Let $q \in \mathbb{N}$, with $q \geq 2$. Then,*

$$\langle a^*_{\beta_1} \dots a^*_{\beta_q} a_{\beta'_1} \dots a_{\beta'_q} \rangle = \lim_{\epsilon\to 0} \frac{(2\epsilon)^{q-1}}{2B} \Big\langle \prod_{j=1}^{q-1} G(k)_{\beta_j\beta'_j} \cdot G^\dagger(k)_{\beta_q\beta'_q} \Big\rangle_k$$

Besides, for any permutation σ of q elements, the equality still holds if, in the right-hand side, all the indices β' are swapped according to σ.

Proof. This theorem easily follows from the definition (2.80) of $G(k)$, the formula (2.82) and the next lemma. The fact that the spectrum of the quantum graph is assumed non-degenerate is crucial for the theorem to hold.

□

Lemma 2.4 *Let $\epsilon > 0$ and let D_ϵ denote the function*

$$D_\epsilon(x) \equiv \frac{1}{1 - e^{i(x+i\epsilon)}}, \quad \forall x \in \mathbb{R}. \tag{2.83}$$

Then, for any integer $q \geq 2$, we have the weak limit

$$\lim_{\epsilon\to 0^+} \frac{(2\epsilon)^{q-1}}{2\pi} D_\epsilon(x)^{q-1} D_\epsilon(-x) = \delta_{2\pi}(x).$$

Proof. Let $f(\epsilon, x)$ be in $L^1(\mathbb{R}, dx)$ for all $\epsilon > 0$ small enough, and let $\varphi(x)$ be a test function. By Plancherel-Parseval,

$$\int dx f(\epsilon, x)\varphi(x) = \frac{1}{\sqrt{2\pi}} \int d\xi \left(\sqrt{2\pi}\hat{f}^*(\epsilon, \xi)\right) \hat{\varphi}(\xi), \tag{2.84}$$

where $\hat{f}$ and $\hat{\varphi}$ denote the Fourier transforms of f and φ respectively. Hence, if $\sqrt{2\pi}\hat{f}(\epsilon, \xi) \to e^{i\xi a}$ as $\epsilon \to 0$, then $f(\epsilon, x) \to \delta(x - a)$. Therefore, to prove the lemma, it suffices to show that

$$I(\epsilon, \xi) \equiv \frac{(2\epsilon)^{q-1}}{2\pi} \int dx\ e^{i\xi x} D_\epsilon(x)^{q-1} D_\epsilon(-x) \to \sum_{p\in\mathbb{Z}} e^{i\xi 2\pi p}. \tag{2.85}$$

In the integrand defining $I(\epsilon, \xi)$, the factor $D_\epsilon(-x)$ brings poles at $x_p^+ = 2\pi p + i\epsilon$ for all $p \in \mathbb{Z}$, and the factor $D_\epsilon(x)^{q-1}$ brings poles at $x_p^- = 2\pi p - i\epsilon$. For $\xi > 0$, Cauchy's residues theorem can be applied, and $I(\epsilon, \xi)$ is determined by the residues at x_p^+. This yields

$$\begin{aligned} I(\epsilon, \xi) &= (2\epsilon)^{q-1} i \sum_{p\in\mathbb{Z}} \mathrm{Res}\left(e^{i\xi x} D_\epsilon(x)^{q-1} D_\epsilon(-x); x_p^+\right) \\ &= (2\epsilon)^{q-1} i \sum_{p\in\mathbb{Z}} \frac{e^{i\xi(2\pi p + i\epsilon)}}{(1 - e^{-2\epsilon})^{q-1}} \lim_{x\to x_p^+} \frac{x - x_p^+}{1 - e^{-i(x - i\epsilon)}} \\ &\to \sum_{p\in\mathbb{Z}} e^{i\xi 2\pi p}, \end{aligned} \tag{2.86}$$

as $\epsilon \to 0$ as wanted. Now, using the same integration method for $\xi < 0$, the residues of the integrand in (2.85) at the poles x_p^- enter into play. These poles are of order $q - 1$. The residue formula gives

$$I(\epsilon, \xi) = \frac{-i(2\epsilon)^{q-1}}{(q-2)!} \sum_{p\in\mathbb{Z}} \frac{d^{q-2}}{dx^{q-2}} \left[\frac{e^{i\xi x}(x - x_p^-)^{q-1}}{(1 - e^{i(x+i\epsilon)})^{q-1}} \cdot D_\epsilon(-x)\right]_{x=x_p^-} \tag{2.87}$$

For all real ϵ, the first factor in the square bracket can be continued at the points $x = x_p^-$ yielding an analytic functions on the balls $B(x_p, 2\pi)$. Hence, taking a derivative with respect to x on this first factor does not increase the degree of singularity at $\epsilon = 0$. By contrast, taking a derivative on the second factor $D_\epsilon(-x)$ does increase the degree of singularity at $\epsilon = 0$. Therefore, in

the limit $\epsilon \to 0$, $I(\epsilon, x)$ is completely determined by the term of (2.87) where the $q-2$ derivatives are performed on the second factor. One finds

$$\begin{aligned}\lim_{\epsilon\to 0} I(\epsilon,\xi) &= \lim_{\epsilon\to 0} \frac{-i^q(2\epsilon)^{q-1}}{(q-2)!} \sum_{p\in\mathbb{Z}} e^{i\xi x_p}(-1)^{q-2}(q-2)! \frac{\left(ie^{-i(x_p-i\epsilon)}\right)^{q-2}}{\left(1-e^{-i(x_p-i\epsilon)}\right)^{q-1}} \\ &= \sum_{p\in\mathbb{Z}} e^{i\xi 2\pi p}, \end{aligned} \tag{2.88}$$

as wanted.

□

In fact, this lemma can easily be generalized to the following result. For any $n_r, n_a \in \mathbb{N}_0$, the weak limit

$$\lim_{\epsilon\to 0^+} \frac{(2\epsilon)^{n_r+n_a+1} n_r! n_a!}{2\pi(n_r+n_a)!} D_\epsilon(x)^{n_r+1} D_\epsilon(-x)^{n_a+1} = \delta_{2\pi}(x) \tag{2.89}$$

holds. The proof is completely similar to the one performed above for the case $n_r = q-2$, $n_a = 0$ in the lemma. Different choices for the natural numbers n_r and n_a with $n_r + n_a = q-2$ lead to different representations of the left-hand side of Theorem 2.3 in terms of products of G and $G^\dagger$.

2.3.2 Alternative Trace Formulae

The Green matrix $G(k)$ can be expanded as the geometric series

$$G(k) = 1 + U_\epsilon(k) + U_\epsilon^2(k) + U_\epsilon^3(k) + \dots \tag{2.90}$$

Together with Theorem 2.3, this expansion easily generates formulae for the autocorrelation functions $C_\beta(q)$ in terms of classical orbits on the graph. In fact, different permutations σ in Theorem 2.3 provide different orbit expansions. For example, it is not difficult to check that the autocorrelation functions of degree 2 can be written

$$\langle |a_\beta|^2 |a_{\beta'}|^2 \rangle = \lim_{\epsilon\to 0} \frac{\epsilon}{B} \sum_{\substack{\vec{\beta}\in C^0 \\ o\vec{\beta}=\beta}} \sum_{\substack{\vec{\beta}'\in C^0 \\ o\vec{\beta}'=\beta'}} e^{-\epsilon(|\vec{\beta}|+|\vec{\beta}'|)} \delta_{l(\vec{\beta}),l(\vec{\beta}')} A_{\vec{\beta}} A^*_{\vec{\beta}'} \tag{2.91}$$

or, if the permutation $\sigma = (1\ 2) \in S_2$ is preferred,

$$\langle |a_\beta|^2 |a_{\beta'}|^2 \rangle = \lim_{\epsilon \to 0} \frac{\epsilon}{B} \sum_{\substack{\vec{\beta}: \\ o\vec{\beta}=\beta, t\vec{\beta}=\beta'}} \sum_{\substack{\vec{\beta}': \\ o\vec{\beta}'=\beta, t\vec{\beta}'=\beta'}} e^{-\epsilon(|\vec{\beta}|+|\vec{\beta}'|)} \delta_{l(\vec{\beta}),l(\vec{\beta}')} A_{\vec{\beta}} A^*_{\vec{\beta}'}. \quad (2.92)$$

Besides, if the variable β' in (2.91) and (2.92) is summed over $\mathbb{N}_{2B}$, two expressions for the mean value $\langle |a_\beta|^2 \rangle$ are obtained. In general, showing the equivalence between expressions obtained from different $\sigma \in S_q$ in Theorem 2.3 turns out to be a difficult problem. Moreover, it can be realized that the trace formula in Subsection 2.1.4 precisely leads to the expression (2.91).

The influence of time-reversal symmetry on eigenvector statistics can be observed on the exact formula (2.91). Indeed, if this symmetry is conserved, and if the paths $\vec{\beta}$ and $\vec{\beta}^r$ are reverse to each other, the stability amplitudes satisfy $A_{\vec{\beta}} = A_{\vec{\beta}^r}$. It follows that β' and $\hat{\beta}'$ lead to the same expression (2.91). Moreover, it is easy to convince oneself that the same is true for autocorrelations of higher degrees. Hence, the eigenvector statistics of a time-reversal invariant graph makes no distinction between an amplitude a_β and the amplitude $a_{\hat{\beta}}$ on the reverse directed bond. This argument confirms the first expectation formulated in 2.2.2.

The orbit interpretation of the Green matrices easily leads to the next following result, which will be of particular importance in the next chapter.

Theorem 2.5 *Let $m \in \mathbb{N}$, and let $\beta_1, \ldots, \beta_m$ and $\beta'_1, \ldots, \beta'_m$ be directed bonds. Then*

$$\left\langle \prod_{j=1}^{m} G(k)_{\beta_j,\beta'_j} \right\rangle_k = \left\langle \prod_{j=1}^{m} G^\dagger(k)_{\beta_j,\beta'_j} \right\rangle_k = \prod_{j=1}^{m} \delta_{\beta_j,\beta'_j}.$$

Proof. From the geometric series (2.90), one has

$$\left\langle \prod_{j=1}^{m} G(k)_{\beta_j,\beta'_j} \right\rangle_k = \left\langle \prod_{j=1}^{m} \sum_{n=0}^{\infty} e^{-\epsilon n} \sum_{\vec{\beta} \in C_n} \delta_{o\vec{\beta},\beta'_j} \delta_{t\vec{\beta},\beta_j} A_{\vec{\beta}} e^{ikl(\vec{\beta})} \right\rangle_k. \quad (2.93)$$

Since the length $l(\vec{\beta})$ are positive if $n = |\vec{\beta}| > 0$, the only term that survives the average over the spectral parameter k is the term coming from $n = 0$ in

each factor j. This precisely gives $\prod_j \delta_{\beta_j,\beta'_j}$. The other equality follows from a similar argument.

□

Third Proof of the Local Weyl Law

The equivalence between the various orbit formulations of the autocorrelation functions can be explicitly checked for the mean values $\langle|a_\beta|^2\rangle$. The expression (2.91), which originates from $\sigma = \mathrm{id} \in S_2$ in Theorem 2.3, is the starting point of the second proof of the local Weyl law presented in Subsection 2.1.4. We will now compute the mean values $\langle|a_\beta|^2\rangle$ by means of (2.92). Establishing the equivalence between the two orbit formulae in this particular case provides at the same time a third proof of the local Weyl law.

First, notice that the expression (2.92) summed over β' enables to write the mean value $\langle|a_\beta|^2\rangle$ as

$$\langle|a_\beta|^2\rangle = \lim_{\epsilon\to 0} \frac{\epsilon}{B} \sum_{n,m=1}^{\infty} e^{-\epsilon(n+m)} Q(n,m), \tag{2.94}$$

where the function $Q(n,m)$ is defined by

$$Q(n,m) \equiv \sum_{\substack{\beta_1,\dots,\beta_{n-1},\beta_n \\ \beta'_1,\dots,\beta'_{m-1}}} S_{\beta,\beta_1}\dots S_{\beta_{n-1},\beta_n} S^*_{\beta,\beta'_1}\dots S^*_{\beta'_{m-1},\beta_n} \delta_{l(\vec{\beta}),l(\vec{\beta}')}. \tag{2.95}$$

In the right-hand side, $\vec{\beta}$ and $\vec{\beta}'$ stand for the paths $(\beta,\beta_1,\dots,\beta_{n-1},\beta_n)$ and $(\beta,\beta'_1\dots,\beta'_{m-1},\beta_n)$. It is easy to see that $Q(1,1) = 1$, and that $Q(n,1)$ and $Q(1,m)$ vanish if $n,m > 1$. In (2.95), the sum over the variable β_n can be performed, and, by unitarity of S, it generates a factor $\delta_{\beta_{n-1},\beta'_{m-1}}$. Therefore, if $n \geq m \geq 2$, $Q(n,m)$ also reads

$$Q(n,m) \equiv \sum_{\substack{\beta_1,\dots,\beta_{n-2},\beta_{n-1} \\ \beta'_1,\dots,\beta'_{m-2}}} S_{\beta,\beta_1}\dots S_{\beta_{n-2},\beta_{n-1}} S^*_{\beta,\beta'_1}\dots S^*_{\beta'_{m-2},\beta_{n-1}} \delta_{l(\vec{\beta}),l(\vec{\beta}')}, \tag{2.96}$$

where now, in each term, the paths $\vec{\beta}$ and $\vec{\beta}'$ refer to $(\beta,\beta_1,\dots,\beta_{n-2},\beta_{n-1})$ and $(\beta,\beta'_1,\dots,\beta'_{m-2},\beta_{n-1})$. By induction, this shows that, for $n \geq m \geq 2$,

$$Q(n,m) = Q(n-1,m-1) = Q(n-m+1,1) = \delta_{n,m}. \tag{2.97}$$

Besides, the same result is obtained for $m \geq n \geq 2$, and hence, $Q(n,m) = \delta_{n,m}$ for all $n, m \geq 1$. This result enables to express (2.94) as

$$\langle |a_\beta|^2 \rangle = \lim_{\epsilon \to 0} \frac{\epsilon}{B} \sum_{n=1}^{\infty} e^{-2\epsilon n} = \lim_{\epsilon \to 0} \frac{\epsilon}{B} \frac{e^{-2\epsilon}}{1 - e^{-2\epsilon}} = \frac{1}{2B}. \tag{2.98}$$

Finally, the local Weyl law (2.13) follows from (2.98) and

$$A_V = \frac{2B}{\mathrm{tr}L} \sum_{\beta=1}^{2B} (VL)_\beta \langle |a_\beta|^2 \rangle. \tag{2.99}$$

Chapter 3

The Generating Function

3.1 Definition and Principles

The Green matrices introduced in the previous chapter can be obtained as derivatives of some determinant. In order to do that, one needs the *retarded* and *advanced source supermatrices* defined by

$$J_r(j_r) \equiv 1 + E_B \otimes j_r \cdot E^{(r)}, \tag{3.1}$$

$$J_a(j_a) \equiv 1 + E_B \otimes j_a \cdot E^{(a)}, \tag{3.2}$$

where E_B is the projector onto the bosonic sector, and

$$j \equiv \begin{pmatrix} j_a \\ j_r \end{pmatrix} \equiv \begin{pmatrix} j_0 \\ j_1 \\ \vdots \\ j_{q-1} \end{pmatrix}, \quad E \equiv \begin{pmatrix} E^{(a)} \\ E^{(r)} \end{pmatrix} \equiv \begin{pmatrix} E^{\alpha_0,\alpha_0} \\ E^{\alpha_1,\alpha_1} \\ \vdots \\ E^{\alpha_{q-1},\alpha_{q-1}} \end{pmatrix}, \tag{3.3}$$

for some integer $q \geq 2$. An introduction to the Bose-Fermi space and to supermatrices is given in Appendix B. The numbers $j_1, \ldots, j_{q-1}$ and j_0, which are contained in j_r and j_a respectively, are called *retarded* and *advanced sources*. They are all required to lie within the unit disc in the complex plane. The number $q-1$ of retarded sources corresponds to the number of matrices E^{α_j,α_j} contained in $E^{(r)}$, so that the product in (3.1) makes sense. These matrices are defined by $(E^{\alpha,\alpha'})_{\beta\beta'} \equiv \delta_{\alpha,\beta}\delta_{\alpha',\beta'}$.

The *generating function* is defined to be

$$\xi_{[\alpha]}(j) \equiv \Big\langle \mathrm{sdet}^{-1}\Big(1 - J_r(j_r)\cdot U_\epsilon(k)\Big)\Big(1 - J_a(j_a)\cdot U_\epsilon^\dagger(k)\Big)\Big\rangle_k . \tag{3.4}$$

The subscript $[\alpha]$ refers to the list of directed bonds $[\alpha] \equiv [\alpha_0, \alpha_1, \ldots, \alpha_{q-1}]$ contained in the source supermatrices, and the variables j are the sources $j \equiv (j_a, j_r) = (j_0, j_1, \ldots, j_{q-1})$.

It will be seen in Chapter 4 that the generating function strongly depends on whether time-reversal symmetry is conserved or broken, and hence this symmetry will play a key role in the forthcoming calculations. It is thus convenient to introduce the time-reversal space TR described in A.2.2. The time-reversal doubling of a matrix $E^{\alpha,\alpha}$ in (3.3) reads

$$\mathcal{E}^{\alpha,\alpha} \equiv \begin{pmatrix} E^{\alpha,\alpha} & 0 \\ 0 & E^{\alpha,\alpha T} \end{pmatrix}_{TR} = \begin{pmatrix} E^{\alpha,\alpha} & 0 \\ 0 & E^{\hat{\alpha},\hat{\alpha}} \end{pmatrix}_{TR} . \tag{3.5}$$

This operation and the resulting algebraic rules are exposed in Appendix A.2.2. We write $\mathcal{E}^{(a)}$ for the time-reversal doubling of $E^{(a)}$, and $\mathcal{E}^{(r)}$ for the column vector containing the time-reversal doublings of the matrices in $E^{(r)}$. The time-reversal doublings of the source supermatrices then become

$$\mathcal{J}_r(j_r) \equiv 1 + E_B \otimes j_r \cdot \mathcal{E}^{(r)}, \tag{3.6}$$

$$\mathcal{J}_a(j_a) \equiv 1 + E_B \otimes j_a \cdot \mathcal{E}^{(a)}, \tag{3.7}$$

and the generating function reads

$$\xi_{[\alpha]}(j) = \mathrm{sdet}^{-1} J_r J_a \Big\langle \mathrm{sdet}^{-\frac{1}{2}}\Big(\mathcal{J}_r^{-1} - \mathcal{U}_\epsilon(k)\Big)\Big(\mathcal{J}_a^{-1} - \mathcal{U}_\epsilon^\dagger(k)\Big)\Big\rangle_k \tag{3.8}$$

Here and in the sequel, the arguments of the source supermatrices are not explicitly written. Notice also that J_r, J_a and their time-reversal doublings are indeed invertible since the sources have moduli smaller than one. It is indeed straightforward to check that

$$J_r^{-1}(j_r) = 1 - E_B \otimes \sum_{k=1}^{n} \frac{j_k}{1+j_k} E^{\alpha_k,\alpha_k} = J_r\left(\frac{-j_k}{1+j_k}\right)_{k\in\mathbb{N}_n}, \tag{3.9}$$

$$J_a^{-1}(j_a) = 1 - E_B \otimes \frac{j_a}{1+j_a} E^{\alpha_0,\alpha_0} = J_a\left(\frac{-j_a}{1+j_a}\right). \tag{3.10}$$

The functions $\xi_{[\alpha]}(j)$ generate the autocorrelation functions of the amplitudes $C_{[\alpha]}$ defined in (2.57) through their derivatives

$$\delta\xi_{[\alpha]} \equiv \prod_{s=0}^{q-1} \frac{\partial}{\partial j_s} \xi_{[\alpha]}(0). \tag{3.11}$$

This is precisely the purpose of the next theorem.

Theorem 3.1 *Let $[\alpha] \equiv [\alpha_0, \alpha_1, \ldots, \alpha_{q-1}]$ be a list of $q \geq 2$ directed bonds. Then, the autocorrelation function $C_{[\alpha]}$ can be written*

$$C_{[\alpha]} = \lim_{\epsilon \to 0} \frac{(2\epsilon)^{q-1}}{2B(q-1)!} \delta\xi_{[\alpha]}.$$

Proof. Let us first compute the derivative of the advanced superdeterminant in $\xi_{[\alpha]}$. By the definition (2.81) of the matrix $G^\dagger(k)$, this derivative can be written

$$\Delta_a(j_a) \equiv \frac{\partial}{\partial j_a} \mathrm{sdet}\Big(1 - J_a U_\epsilon^\dagger(k)\Big)^{-1} = \frac{\partial}{\partial j_a} \det\Big(1 - j_a E^{\alpha_0,\alpha_0}\big(G^\dagger(k) - 1\big)\Big)^{-1}. \tag{3.12}$$

Making use of Theorem C.1, $\Delta_a \equiv \Delta_a(0)$ becomes

$$\Delta_a = \mathrm{tr} E^{\alpha_0,\alpha_0}\big(G^\dagger(k) - 1\big) = \big(G^\dagger(k) - 1\big)_{\alpha_0,\alpha_0}. \tag{3.13}$$

The same operation has now to be performed on the retarded superdeterminant of $\xi_{[\alpha]}$. One gets

$$\begin{aligned} \Delta_r(j_r) &\equiv \prod_{s=1}^{q-1} \frac{\partial}{\partial j_s} \mathrm{sdet}\Big(1 - J_r(j) U_\epsilon(k)\Big)^{-1} \\ &= \prod_{s=1}^{q-1} \frac{\partial}{\partial j_s} \det\Big(1 - \sum_{s=1}^{q-1} j_s E^{\alpha_s,\alpha_s}\big(G(k) - 1\big)\Big)^{-1}. \end{aligned} \tag{3.14}$$

The quantity $\Delta_r \equiv \Delta_r(0)$ is obtained as previously by applying Theorem C.1 to the last expression. It reads

$$\begin{aligned} \Delta_r &= \sum_{\sigma \in S_{q-1}} \prod_{s=1}^{q-1} \sum_{\gamma_s=1}^{2B} \Big[E^{\alpha_s,\alpha_s}\big(G(k) - 1\big)\Big]_{\gamma_s,\gamma_{\sigma(s)}} \\ &= \sum_{\sigma \in S_{q-1}} \prod_{s=1}^{q-1} \big(G(k) - 1\big)_{\alpha_s,\alpha_{\sigma(s)}}. \end{aligned} \tag{3.15}$$

One is now ready to bring the advanced and retarded parts together and compute the limit $\epsilon \to 0$ of

$$\begin{aligned} \frac{(2\epsilon)^{q-1}}{2B}\delta\xi_{[\alpha]} &= \frac{(2\epsilon)^{q-1}}{2B}\langle \Delta_a \Delta_r \rangle_k \\ &= \frac{(2\epsilon)^{q-1}}{2B} \sum_{\sigma \in S_{q-1}} \left\langle \left(G^\dagger(k)-1\right)_{\alpha_0,\alpha_0} \prod_{s=1}^{q-1} \left(G(k)-1\right)_{\alpha_s,\alpha_{\sigma(s)}} \right\rangle_k . \end{aligned} \tag{3.16}$$

By Theorem 2.5, a product of components of $G(k)-1$ has a vanishing spectral average, and similarly with a product containing elements of $G^\dagger(k)-1$ only. This implies that the matrix $G^\dagger - 1$ in (3.16) can merely be replaced with $G^\dagger$. This also implies that the term coming from choosing the matrix -1 in each of the $q-1$ factors $G-1$ does not contribute. Let us now consider a term of (3.16) containing $G^\dagger{}_{\alpha_0,\alpha_0}$ and p components of G for some $1 \leq p \leq q-1$. Theorem 2.3 states that the spectral average of such a term has a pole at $\epsilon = 0$ and that the Laurent series starts with ϵ^{-p}. Therefore, in the limit of vanishing ϵ, the prefactor ϵ^{q-1} in (3.16) kills all the terms containing less than $q-1$ components of G, and one remains with

$$\lim_{\epsilon\to 0} \frac{(2\epsilon)^{q-1}}{2B}\delta\xi_{[\alpha]} = \sum_{\sigma \in S_{q-1}} \lim_{\epsilon\to 0} \frac{(2\epsilon)^{q-1}}{2B} \left\langle G^\dagger(k)_{\alpha_0,\alpha_0} \prod_{s=1}^{q-1} G(k)_{\alpha_s,\alpha_{\sigma(s)}} \right\rangle_k . \tag{3.17}$$

Theorem 2.3 also claims that, in an expression of this type, each permutation σ contributes exactly the same amount. Hence,

$$\lim_{\epsilon\to 0} \frac{(2\epsilon)^{q-1}}{2B(q-1)!}\delta\xi_{[\alpha]} = \lim_{\epsilon\to 0} \frac{(2\epsilon)^{q-1}}{2B} \left\langle G^\dagger(k)_{\alpha_0,\alpha_0} \prod_{s=1}^{q-1} G(k)_{\alpha_s,\alpha_s} \right\rangle_k . \tag{3.18}$$

Moreover, the same theorem also gives the result of such a limit. It reads

$$\lim_{\epsilon\to 0} \frac{(2\epsilon)^{q-1}}{2B(q-1)!}\delta\xi_{[\alpha]} = \left\langle \prod_{s=0}^{q-1} |a_{\alpha_s}|^2 \right\rangle . \tag{3.19}$$

Finally, notice that the right-hand side of this chain of equalities is nothing else but the definition (2.57) of $C_{[\alpha]}$. This concludes the proof of the theorem.

□

Property 3.2 *For all j_a and $j_r = (j_1, \ldots, j_{q-1})$ in a sufficiently small neighborhood of the origin,*

$$\xi_{[\alpha]}(j_a, 0) = \xi_{[\alpha]}(0, j_r) = 1.$$

Proof. We show here that $\xi_{[\alpha]}(0, j_r) = 1$, the proof of the second equality being totally similar. From the definition (3.4) of $\xi_{[\alpha]}$, it is obvious that $\xi_{[\alpha]}(0, 0) = 1$. Besides, the function

$$\xi_{[\alpha]}(0, j_r) = \left\langle \det\left(1 - j_r \cdot E^{(r)}(G(k) - 1)\right)^{-1} \right\rangle_k \tag{3.20}$$

only has point singularities for $j_r \in \mathbb{C}^{q-1}$ and cannot have poles accumulating at the origin since the limit $j_r \to 0$ exists. Therefore, $\xi_{[\alpha]}(0, j_r)$ is analytic in a neighborhood of the origin. It is thus sufficient to show that, for all $p = (p_1, \ldots, p_{q-1})$ in $\mathbb{N}_0^{q-1}$,

$$\Delta_p \equiv \prod_{k=1}^{q-1} \frac{\partial^{p_k}}{\partial j_k^{p_k}} \xi_{[\alpha]}(0, 0) = 0 \tag{3.21}$$

if $p \equiv \sum_{k=1}^{q-1} p_k \geq 1$. But, it is easy to see that Δ_p can also be written

$$\Delta_p = \prod_{k=1}^{p} \frac{\partial}{\partial j'_k} \xi_{[\alpha']}(0, 0) = 0, \tag{3.22}$$

where $[\alpha'] \equiv [\alpha_0, \alpha'_1, \ldots, \alpha'_p]$ is the list of directed bonds obtained from the initial list $[\alpha] \equiv [\alpha_0, \alpha_1, \ldots, \alpha_{q-1}]$ by repeating each element α_s in $[\alpha]$ p_s times for all $s \in \mathbb{N}_{q-1}$. Hence, Δ_p is nothing else but the quantity Δ_r defined in (3.14), and the formula (3.15) applied to $[\alpha']$ yields

$$\Delta_p = \sum_{\sigma \in S_p} \left\langle \prod_{s=1}^{p} (G(k) - 1)_{\alpha'_s, \alpha'_{\sigma(s)}} \right\rangle_k \tag{3.23}$$

Then, Theorem 2.5 ensures that Δ_p vanishes. This proves the property.

□

3.2 Diagonal Approximation

In the subsections 2.1.4 and 2.3.2, the autocorrelation functions $C_{[\alpha,\alpha']}$ of degree 2 have been written in terms of classical orbits on the graph. In particular, the formula (2.47) for the fluctuations $\mathcal{F}_V$ has been obtained from a diagonal approximation on the set of long primitive orbits. We show here that taking the diagonal approximation on the generating function and then computing the derivatives as prescribed in Theorem 3.1 leads to a different formula.

The first step is to notice that, by definition of $\xi_{[\alpha,\alpha']}(j_a, j_r)$, and by Theorem 3.1, the autocorrelation functions $C_{[\alpha,\alpha']}$ of degree 2 can also be written

$$C_{[\alpha,\alpha']} = \lim_{\epsilon\to 0} \frac{\epsilon}{B}\frac{\partial}{\partial j_r}\frac{\partial}{\partial j_a}\Xi_{[\alpha,\alpha']}(0,0), \tag{3.24}$$

where the function $\Xi_{[\alpha,\alpha']}(j_a, j_r)$ introduced reads

$$\Xi_{[\alpha,\alpha']}(j_a, j_r) \equiv \Big\langle \log\det\Big(1 - J_r(j_r)_{BB}U_\epsilon(k)\Big)\log\det\Big(1 - J_a(j_a)_{BB}U_\epsilon^\dagger(k)\Big)\Big\rangle_k \tag{3.25}$$

and $J_r(j_r)_{BB}$ and $J_a(j_a)_{BB}$ are the Bose-Bose blocks of the supermatrices in (3.1) and (3.2) built from the projectors $E^{(a)} = E^{\alpha,\alpha}$ and $E^{(r)} = E^{\alpha',\alpha'}$.

The formula $\log\det = \mathrm{tr}\log$ enables to write the new generating function (3.25) in terms of closed paths on the graph. Indeed, expanding the logarithms gives

$$\begin{aligned}\Xi_{[\alpha,\alpha']}(j_a, j_r) &= \sum_{n,m=0}^{\infty}\frac{1}{nm}\Big\langle \mathrm{tr}\Big(J_r(j_r)_{BB}U_\epsilon(k)\Big)^n \mathrm{tr}\Big(J_a(j_a)_{BB}U_\epsilon^\dagger(k)\Big)^m\Big\rangle_k \\ &= \sum_{n,m=0}^{\infty}\frac{1}{nm}\sum_{\vec\beta\in C_n^0}\sum_{\vec\beta'\in C_m^0} A_{\epsilon,\vec\beta}(j_r)A^*_{\epsilon,\vec\beta'}(j_a)\delta_{l(\vec\beta),l(\vec\beta')}\end{aligned} \tag{3.26}$$

with the modified stability amplitudes $A_{\epsilon,\vec\beta}(j_r)$ and $A_{\epsilon,\vec\beta}(j_a)$ defined by

$$A_{\epsilon,\vec\beta}(j_r) \equiv \sum_{i=0}^{n-1}\big[J_r(j_r)_{BB}S_\epsilon\big]_{\beta_{i+1},\beta_i} \quad\text{and}\quad A_{\epsilon,\vec\beta}(j_a) \equiv \sum_{i=0}^{n-1}\big[J_a(j_a)_{BB}S_\epsilon\big]_{\beta_{i+1},\beta_i} \tag{3.27}$$

for any classical path $\vec\beta = (\beta_0, \beta_1, \ldots, \beta_n)$. In terms of periodic orbits, the

formula (3.26) becomes

$$\begin{aligned}\Xi_{[\alpha,\alpha']}(j_a,j_r) &= \sum_{p,p'} \frac{A_{\epsilon,p}(j_r)A^*_{\epsilon,p'}(j_a)}{|p||p'|}\frac{|p||p'|}{r_p r_{p'}}\delta_{l_p,l_{p'}} \\ &= \sum_{p,p'} \frac{A_{\epsilon,p}(j_r)A^*_{\epsilon,p'}(j_a)}{r_p r_{p'}}\delta_{l_p,l_{p'}}. \end{aligned} \tag{3.28}$$

Then, the diagonal approximation is taken on the last expression, and the non-primitive periodic orbits are neglected. The *diffusion* generating function $\Xi^{\mathrm{diag},D}_{[\alpha,\alpha']}(j_a,j_r)$ is defined by only retaining the pairs of periodic orbits $p'=p$ in (3.28), and the *cooperon* generating function $\Xi^{\mathrm{diag},C}_{[\alpha,\alpha']}(j_a,j_r)$ is defined by considering the pairs $p'=\hat{p}$. This terminology is motivated by an analogy with the results found in Chapter 5. The diagonal approximation thus leads to the formulae

$$\Xi^{\mathrm{diag}}_{[\alpha,\alpha']}(j_a,j_r) = \begin{cases} \Xi^{\mathrm{diag},D}_{[\alpha,\alpha']}(j_a,j_r) & (U) \\ \Xi^{\mathrm{diag},D}_{[\alpha,\alpha']}(j_a,j_r) + \Xi^{\mathrm{diag},C}_{[\alpha,\alpha']}(j_a,j_r) & (O) \end{cases} \tag{3.29}$$

for the unitary (U) and orthogonal (O) classes. Besides, one can come back to expressions in terms of closed paths and write

$$\Xi^{\mathrm{diag},D}_{[\alpha,\alpha']}(j_a,j_r) = \sum_p A_{\epsilon,p}(j_r)A^*_{\epsilon,p}(j_a) = \sum_{\vec{\beta}\in C^0} \frac{A_{\epsilon,\vec{\beta}}(j_r)A^*_{\epsilon,\vec{\beta}}(j_a)}{|\vec{\beta}|} \tag{3.30}$$

$$\Xi^{\mathrm{diag},C}_{[\alpha,\alpha']}(j_r,j_a) = \sum_p A_{\epsilon,p}(j_r)A^*_{\epsilon,\hat{p}}(j_a) = \sum_{\vec{\beta}\in C^0} \frac{A_{\epsilon,\vec{\beta}}(j_r)A^*_{\epsilon,\vec{\beta}^r}(j_a)}{|\vec{\beta}|} \tag{3.31}$$

where $\vec{\beta}^r$ denotes the reverse of $\vec{\beta}$. Finally, the formulae in the right-hand sides can be interpreted as the trace of logarithms of some matrices. For this purpose, let us introduce the diffusion and cooperon modified classical maps $M^D_\epsilon(j_a,j_r)$ and $M^C_\epsilon(j_a,j_r)$ by

$$\begin{aligned} M^D_\epsilon(j_a,j_r)_{\beta_1,\beta_2} &\equiv \left[J_r(j_r)_{BB}S_\epsilon\right]_{\beta_1,\beta_2}\left[J_a(j_a)_{BB}S^*_\epsilon\right]_{\beta_1,\beta_2} \\ &= J_r(j_r)_{BB,\beta_1\beta_1}J_a(j_a)_{BB,\beta_1\beta_1}M_{\epsilon,\beta_1,\beta_2} \end{aligned} \tag{3.32}$$

$$\begin{aligned} M^C_\epsilon(j_a,j_r)_{\beta_1,\beta_2} &\equiv \left[J_r(j_r)_{BB}S_\epsilon\right]_{\beta_1,\beta_2}\left[J_a(j_a)_{BB}S^*_\epsilon\right]_{\beta_2,\beta_1} \\ &= J_r(j_r)_{BB,\beta_1\beta_1}J_a(j_a)_{BB,\beta_2\beta_2}M_{\epsilon,\beta_1,\beta_2} \end{aligned} \tag{3.33}$$

In (3.33), $S^T = S$ has been supposed. This can be done since the cooperon quantities only occur in the time-reversal invariant case. With these definitions, (3.30) and (3.31) become

$$\Xi^{\mathrm{diag},D}_{[\alpha,\alpha']}(j_a, j_r) = -\log\det\left(1 - M^D_\epsilon(j_a, j_r)\right) \tag{3.34}$$

$$\Xi^{\mathrm{diag},C}_{[\alpha,\alpha']}(j_a, j_r) = -\log\det\left(1 - M^C_\epsilon(j_a, j_r)\right) \tag{3.35}$$

The derivatives with respect to the sources j_r and j_a can now be performed on (3.34) and (3.35). Let us first take the derivative with respect to j_a. By Theorem C.4, one gets

$$\frac{\partial \Xi^{\mathrm{diag},\circ}_{[\alpha,\alpha']}}{\partial j_a}(0, j_r) = \mathrm{tr}\left(M^\circ_{\epsilon,a}(0, j_r)\frac{1}{1 - M^\circ_\epsilon(0, j_r)}\right) \tag{3.36}$$

for $\circ \in \{D, C\}$, where $M^\circ_{\epsilon,a}$ stands for the derivative of M°_ϵ with respect to j_a. Then, the derivative with respect to j_r yields

$$\frac{\partial^2 \Xi^{\mathrm{diag},\circ}_{[\alpha,\alpha']}}{\partial j_r \partial j_a}(0,0) = \mathrm{tr}\left(M^\circ_{\epsilon,r,a}\frac{1}{1 - M_\epsilon} + M^\circ_{\epsilon,a}\frac{1}{1 - M_\epsilon}M^\circ_{\epsilon,r}\frac{1}{1 - M_\epsilon}\right) \tag{3.37}$$

where all the matrix-valued functions in the trace of the right-hand side are evaluated at $(j_a, j_r) = (0, 0)$. Besides, the definition (3.32) gives

$$M^D_{\epsilon,r,a} = \delta_{\alpha,\alpha'}E^{\alpha,\alpha}M_\epsilon, \quad M^\circ_{\epsilon,r} = E^{\alpha',\alpha'}M_\epsilon, \quad M^\circ_{\epsilon,a} = E^{\alpha,\alpha}M_\epsilon. \tag{3.38}$$

Therefore,

$$\frac{\partial^2 \Xi^{\mathrm{diag},D}_{[\alpha,\alpha']}}{\partial j_r \partial j_a}(0,0) = \delta_{\alpha,\alpha'}\left(\frac{M_\epsilon}{1 - M_\epsilon}\right)_{\alpha,\alpha} + \left(\frac{M_\epsilon}{1 - M_\epsilon}\right)_{\alpha,\alpha'}\left(\frac{M_\epsilon}{1 - M_\epsilon}\right)_{\alpha',\alpha}. \tag{3.39}$$

The last step of the calculation consists in multiplying (3.39) by $\frac{\epsilon}{B}$ and in extracting the lowest order terms in ϵ. This procedure generates $C^{\mathrm{diag},D}_{\epsilon,[\alpha,\alpha']}$. By analogy with (3.24), one expects this quantity to have a positive limit as $\epsilon \to 0$. One can make use of Corollary 1.2 in order to extract the small ϵ dependency of the matrix $M_\epsilon(1 - M_\epsilon)^{-1}$ in (3.39). Indeed, this corollary expresses this matrix as a sum of an uniform contribution $|1\rangle\langle 1|$, which contains the whole singularity in ϵ, and a massive part R. After some algebra, one gets

$$C^{\mathrm{diag},D}_{[\alpha,\alpha']} = \frac{1}{(2B)^2}\left[\delta_{\alpha,\alpha'} + R_{\alpha\alpha'} + R_{\alpha'\alpha} + \frac{1}{2B}\left(\frac{1}{2\epsilon} - 1\right)\right]. \tag{3.40}$$

The last term in the square bracket has a non-expected singularity at $\epsilon = 0$. This divergence originates from taking the uniform part $|1\rangle\langle 1|$ of $M_\epsilon(1-M_\epsilon)^{-1}$ in both factors of the second term in (3.39).

If the graph is in the unitary symmetry class, the cooperon quantities are not taken into account. However, if the graph is time-reversal invariant, repeating the procedure above with the cooperon quantities yields an autocorrelation $C^{\mathrm{diag},C}_{\epsilon,[\alpha,\alpha']}$ similar to (3.40) with α replaced with $\hat{\alpha}$. The preceding formulae show that, within the diagonal approximation scheme presented in this section, the fluctuations of any observable V read

$$\mathcal{F}_V^{\mathrm{diag}} = \left[\infty + \kappa\frac{\mathrm{tr}(VL)^2}{(\mathrm{tr}L)^2}\right] + 2\kappa\frac{\sum_{\alpha,\alpha'}[VL\cdot R\cdot VL]_{\alpha,\alpha'}}{(\mathrm{tr}L)^2}. \tag{3.41}$$

Here, the symbol ∞ refers to the diverging term of (3.40). In fact, in addition to this divergence, the universal term in the square brackets is not realistic at all since, for an observable with $\bar{V} \neq 0$, the first term should be $(\mathrm{tr}VL)^2$ as in (2.76). It is important to notice that this universal term is composed of all the contributions obtained from (3.37) by systematically replacing $M_\epsilon(1-M_\epsilon)^{-1}$ with its uniform component $|1\rangle\langle 1|$. We deduce that this approximation does not handle the component $|1\rangle\langle 1|$ of $M_\epsilon(1-M_\epsilon)^{-1}$ properly.

Notice that the diagonal long orbits in Subsection 2.1.4 leads to well-behaved universal fluctuations. In the chapters 4 and 5, the more elaborated supersymmetry method will show that both these diagonal approximations capture a part of the truth. The universal contribution to the fluctuations is indeed given by the long diagonal orbits formula, whereas the first system-dependent corrections are those obtained in (3.41).

3.3 Nonlinear Supersymmetric σ Model

In this section, an exact representation of the generating function $\xi_{[\alpha]}(j)$ in terms of a 0-dimensional nonlinear supersymmetric σ-model is built. The approach follows in the main lines the derivation in [40] and [41] of a similar

model used for the investigation of the eigenvalue statistics on quantum graph.

By the results exposed in Appendix D concerning determinants of block matrices, the retarded and advanced factors in (3.8) read

$$\mathrm{sdet}^{-1/2}(\mathcal{J}_r^{-1} - \mathcal{U}_\epsilon) = \mathrm{sdet}^{-1/2} \begin{pmatrix} 1 & \sqrt{\mathcal{S}_\epsilon} T \\ T\sqrt{\mathcal{S}_\epsilon} & \mathcal{J}_r^{-1} \end{pmatrix}, \tag{3.42}$$

and

$$\mathrm{sdet}^{-1/2}(\mathcal{J}_a^{-1} - \mathcal{U}_\epsilon^\dagger) = \mathrm{sdet}^{-1/2} \begin{pmatrix} 1 & \sqrt{\mathcal{S}_\epsilon}^\dagger T^\dagger \\ T^\dagger\sqrt{\mathcal{S}_\epsilon}^\dagger & \mathcal{J}_a^{-1} \end{pmatrix}. \tag{3.43}$$

In these expressions, the square root of a matrix is obtained by keeping the same eigenvectors and by taking the square roots of the eigenvalues. In this definition, it is important to always keep the same convention for the square roots of the eigenvalues. One can for example choose the eigenphases in $(-\pi, \pi]$, fixing in this way the half-line singularity of the square root to $(-\infty, 0]$. Then, it is not difficult to see that this definition leads to the natural properties $\sqrt{A}\sqrt{A} = A$ and $\sqrt{A^\dagger} = \sqrt{A}^\dagger$. These two properties have been used in order to obtain (3.42) and (3.43). Besides, if $\mathcal{A}$ is the time-reversal doubled of A, then $\sqrt{\mathcal{A}}$ is the time-reversal doubled of $\sqrt{A}$.

Lemma 3.3 *Let $\mathcal{A}_r(j_r)$ and $\mathcal{A}_a(j_a)$ denote the supermatrices in the right-hand sides of (3.42) and (3.43). Then,*

- *For $j \in \mathbb{R}^{n+1}$ sufficiently close to the origin, the eigenvalues of the Bose-Bose blocks $\mathcal{A}_{rBB}(j_r)$ and $\mathcal{A}_{aBB}(j_a)$ all have positive real parts.*

- *$\mathcal{A}_r(j_r)$ and $\mathcal{A}_a(j_a)$ are obtained from their components $(\uparrow, \uparrow)$ in time-reversal space by time-reversal doubling.*

Proof. We give the proofs concerning the supermatrix $\mathcal{A}_r(j_r)$, the arguments for $\mathcal{A}_a(j_a)$ being identical. In order to prove the first assertion of the lemma, it is sufficient to prove the wanted property for the matrix $\mathcal{A}_{rBB}(0)$. Then,

one concludes by continuity. The characteristic polynomial of $\mathcal{A}_{r\mathrm{BB}}(0)$ reads

$$\begin{aligned}\det\begin{pmatrix} 1-\lambda & \sqrt{S_\epsilon}T \\ T\sqrt{S_\epsilon} & 1-\lambda \end{pmatrix} &= \det\left((1-\lambda)^2 - U_\epsilon\right) \\ &= \det\left((1-\lambda)^2 - U_\epsilon\right)^2. \end{aligned} \tag{3.44}$$

To obtain the first equality the results exposed in Appendix D have been used. If $\theta_j(k)$ denote the $2B$ eigenphases of $U(k)$, this polynomial becomes

$$\left(\prod_{j=1}^{2B}\left((1-\lambda)^2 - e^{i\theta_j(k)-\epsilon}\right)\right)^2 \tag{3.45}$$

which vanishes if and only if it exists $j \in \mathbb{N}_{2B}$ such that $(1-\lambda)^2 = e^{i\theta_j(k)-\epsilon}$, that is

$$\lambda = 1 \pm e^{i\theta_j(k)-\epsilon}. \tag{3.46}$$

Since $\epsilon > 0$ by assumption, the eigenvalues of $\mathcal{A}_r$ all have positive real parts.

The second point of the lemma is straightforward since $\mathcal{J}_r$ is obtained from J_r by time-reversal doubling and since $\sqrt{S_\epsilon}$ is the time-doubled of $\sqrt{S_\epsilon}$.

□

One can make use of the formulae (3.42) and (3.43) and of the last lemma to express the generating function $\xi_{[\alpha]}(j)$ as a Gaussian superintegral. Appendix B provides a short introduction to these techniques. This procedure requires the prior introduction of a D-dimensional complex Grassmann algebra Λ, where $D = 8B$ is the dimension of the $\mathbb{C}$-linear space $V \equiv \mathcal{A} \otimes RA \otimes X$. In this notation, X stands for the auxiliary space introduced in (3.42) and (3.43), and the retarded-advanced space, denoted by RA, originates from melting together the retarded sector in (3.42) and the advanced sector in (3.43). Both these spaces are 2-dimensional $\mathbb{C}$-linear space. The formula (3.8) for $\xi_{[\alpha]}(j)$ can then be written as

$$\xi_{[\alpha]}(j) = \mathrm{sdet}^{-1} J_r J_a \cdot \int d\psi \left\langle e^{-S[\Psi]} \right\rangle_k. \tag{3.47}$$

for some quadratic form $S[\Psi]$. The integration is performed using Berezin's rules on the Grassmann envelope $(V \oplus V)(\Lambda)$. The integration variable ψ in (3.47) is thus a supervector containing $8B$ commuting complex components, each of which is integrated over the whole complex plane, and $8B$ anticommuting ones, integrated according to Berezin's rules. The symbol Ψ refers to the time-doubled of ψ, and $\bar{\Psi}$ refers to its dual, that is

$$\Psi \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \psi \\ \sigma_1^d \psi^* \end{pmatrix} \quad \text{and} \quad \bar{\Psi} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \psi^\dagger & \psi \sigma_1^d \sigma_3^{BF} \end{pmatrix}. \tag{3.48}$$

More details about time-reversal doublings of vectors and matrices and the resulting calculus are given in Appendix A. The function $S[\Psi]$ in the formula (3.47) reads

$$\begin{aligned} S[\Psi] \equiv & \begin{pmatrix} \bar{\Psi}_{r1} & \bar{\Psi}_{r2} \end{pmatrix} \begin{pmatrix} 1 & \sqrt{\mathcal{S}_\epsilon} T \\ T\sqrt{\mathcal{S}_\epsilon} & \mathcal{J}_r^{-1} \end{pmatrix} \begin{pmatrix} \Psi_{r1} \\ \Psi_{r2} \end{pmatrix} \\ & + \begin{pmatrix} \bar{\Psi}_{a1} & \bar{\Psi}_{a2} \end{pmatrix} \begin{pmatrix} 1 & \sqrt{\mathcal{S}_\epsilon}^\dagger T^\dagger \\ T^\dagger \sqrt{\mathcal{S}_\epsilon}^\dagger & \mathcal{J}_a^{-1} \end{pmatrix} \begin{pmatrix} \Psi_{a1} \\ \Psi_{a2} \end{pmatrix}. \end{aligned} \tag{3.49}$$

In this definition and in the sequel, for any expression involving the RA space, the indices r and a indicate the retarded and advanced components respectively. The indices 1 and 2 refer to the first and second components in the auxiliary space X. The function S can be decomposed as the sum $S = S_0 + S_{cf}$, where

$$S_0[\Psi] \equiv \bar{\Psi}_{r1}\Psi_{r1} + \bar{\Psi}_{a1}\Psi_{a1} + \bar{\Psi}_{r2}\mathcal{J}_r^{-1}\Psi_{r2} + \bar{\Psi}_{a2}\mathcal{J}_a^{-1}\Psi_{a2}, \tag{3.50}$$

is the k-independent part, and

$$\begin{aligned} S_{cf}[\Psi] \equiv & \ \bar{\Psi}_{r1}\sqrt{\mathcal{S}_\epsilon}T\Psi_{r2} + \bar{\Psi}_{r2}T\sqrt{\mathcal{S}_\epsilon}\Psi_{r1} \\ & + \bar{\Psi}_{a1}\sqrt{\mathcal{S}_\epsilon}^\dagger T^\dagger \Psi_{a2} + \bar{\Psi}_{a2}T^\dagger \sqrt{\mathcal{S}_\epsilon}^\dagger \Psi_{a1} \qquad (3.51) \\ = & \ 2\bar{\Psi}_{r1}\sqrt{\mathcal{S}_\epsilon}T\Psi_{r2} + 2\bar{\Psi}_{a2}T^\dagger\sqrt{\mathcal{S}_\epsilon}^\dagger\Psi_{a1}. \qquad (3.52) \end{aligned}$$

is k-dependent. To get the last equality, the invariance of T and $\sqrt{\mathcal{S}_\epsilon}$ under the generalized transposition defined in (A.51) has been used.

The next step consists in performing the spectral average. In [6], it is shown that, provided the bond lengths $\{L_b\}_{b\in\mathbb{N}_B}$ are incommensurate, the identity

$$\langle f(kL_1,\ldots,kL_B)\rangle_k = \int_{[0,2\pi]^B} \frac{d\varphi_1\ldots d\varphi_B}{(2\pi)^B} f(\varphi_1,\ldots,\varphi_B) \tag{3.53}$$

holds for any integrable function f on the B-torus. As a consequence, the exponential of $S_{cf}[\Psi]$ in (3.52) has the spectral average

$$\left\langle e^{-S_{cf}[\Psi]}\right\rangle_k = \prod_{b=1}^{B}\int_0^{2\pi}\frac{d\varphi_b}{2\pi}e^{-S^b_{cf}[\Psi_b;\varphi_b]}, \tag{3.54}$$

where

$$S^b_{cf}[\Psi_b;\varphi_b] \equiv 2\sum_{d=\pm}\left[\left(\bar\Psi_{r1}\sqrt{\mathcal{S}_\epsilon}\right)_{bd}e^{i\varphi_b}\Psi_{r2;bd} + \bar\Psi_{a2;bd}e^{-i\varphi_b}\left(\sqrt{\mathcal{S}_\epsilon}^{\dagger}\Psi_{a1}\right)_{bd}\right] \tag{3.55}$$

Let us state a general theorem, called color-flavor transformation, proved by M.R. Zirnbauer in [76]. Let $\phi_1,\ldots,\phi_4$ be four supervectors in $(W'\otimes W')(\Lambda)$, where $W=\mathbb{C}^n\otimes\mathbb{C}^m$. For $n\in\mathbb{N}_4$, $\phi^j_{n,\alpha}$ denotes the supervector in $\Lambda^{|1|}$ containing the components $(j,\alpha)\in\mathbb{N}_n\times\mathbb{N}_m$ of ϕ_n, and the summation convention over repeated indices $j\in\mathbb{N}_n$ or $\alpha\in\mathbb{N}_m$ is adopted. Then, if $d\mu_H$ stands for the Haar measure on $U(n)$, the *color-flavor* identity reads

$$\begin{aligned}&\int_{U(n)} d\mu_H(U)e^{-\phi^{*j}_{1,\alpha}U^{j,k}\phi^k_{2,\alpha}-\phi^{*j}_{3,\alpha}U^{\dagger j,k}\phi^k_{4,\alpha}}\\ &= \int d(Z,\tilde Z)\mathrm{sdet}(1-Z\tilde Z)e^{-\phi^{*j}_{1,\alpha}Z_{\alpha,\beta}\phi^j_{4,\beta}-\phi^{*j}_{3,\alpha}\tilde Z_{\alpha,\beta}\phi^j_{2,\beta}}.\end{aligned} \tag{3.56}$$

In the right-hand side, the supermatrix variables Z and $\tilde Z$ are in $L(\mathbb{C}^m|\mathbb{C}^m)$, and the integration is performed over all such supermatrices satisfying

$$\tilde Z_{BB} = Z^\dagger_{BB},\quad \tilde Z_{FF} = -Z^\dagger_{FF}, \tag{3.57}$$

and such that the eigenvalues of the positive hermitian matrix $Z^\dagger_{BB}Z_{BB}$ are less than unity.

The color-flavor transformation (3.56) can be applied to replace the average over the B-torus in (3.54) with a superintegral with respect to a supermatrix

field. Indeed, the integration measure in (3.54) is B times the Haar measure on $U(1)$. Hence,

$$\left\langle e^{-S_{cf}[\Psi]}\right\rangle_k = \prod_{b=1}^{B} \int d(Z_b, \tilde{Z}_b) \mathrm{sdet}(1 - Z_b\tilde{Z}_b) e^{-S^b_{cf}[\Psi_b; Z_b, \tilde{Z}_b]}, \tag{3.58}$$

with

$$\begin{aligned} S^b_{cf}[\Psi_b; Z_b, \tilde{Z}_b] &= 2\sum_{d=\pm} \left(\bar{\Psi}_{r1}\sqrt{\mathcal{S}_\epsilon}\right)_{bd} Z_{b;dd'} \left(\sqrt{\mathcal{S}_\epsilon}^{\dagger}\Psi_{a1}\right)_{bd'} \\ &\quad +2\sum_{d=\pm} \bar{\Psi}_{a2;bd}\tilde{Z}_{b,dd'}\Psi_{r2;bd} \end{aligned} \tag{3.59}$$

Each matrix Z_b or $\tilde{Z}_b$ lies in $L(\mathcal{A}_d \otimes TR | \mathcal{A}_d \otimes TR)$, where $\mathcal{A}_d$ is the 2-dimensional direction space. In order to simplify the notations, one can introduce the new supermatrix fields

$$Z = \bigoplus_{b=1}^{B} Z_b \quad \text{and} \quad \tilde{Z} = \bigoplus_{b=1}^{B} \tilde{Z}_b, \tag{3.60}$$

which belong to $L(\mathcal{A} \otimes TR | \mathcal{A} \otimes TR)$. It is easy to check that Z and $\tilde{Z}$ still satisfy the color-flavor requirements (3.57). Then, the formula (3.58) becomes

$$\left\langle e^{-S_{cf}[\Psi]}\right\rangle_k = \int d(Z, \tilde{Z}) \mathrm{sdet}(1 - Z\tilde{Z}) e^{-S_{cf}[\Psi; Z, \tilde{Z}]}, \tag{3.61}$$

where now, in addition to the color-flavor conditions (3.57), the measure $d(Z, \tilde{Z})$ also imposes to the variables Z and $\tilde{Z}$ to be diagonal in the bond space $\mathcal{A}_b$, and

$$\begin{aligned} S_{cf}[\Psi; Z, \tilde{Z}] &= 2\bar{\Psi}_{r1}\sqrt{\mathcal{S}_\epsilon} Z \sqrt{\mathcal{S}_\epsilon}^{\dagger}\Psi_{a1} + 2\bar{\Psi}_{a2}\tilde{Z}\Psi_{r2} \\ &= \bar{\Psi}_{r1}\sqrt{\mathcal{S}_\epsilon} Z \sqrt{\mathcal{S}_\epsilon}^{\dagger}\Psi_{a1} + \bar{\Psi}_{a1}\sqrt{\mathcal{S}_\epsilon}^{\dagger} Z^{\tau} \sqrt{\mathcal{S}_\epsilon}\Psi_{r1} \\ &\quad + \bar{\Psi}_{a2}\tilde{Z}\Psi_{r2} + \bar{\Psi}_{r2}\tilde{Z}^{\tau}\Psi_{a2}. \end{aligned} \tag{3.62}$$

The formula (3.62), together with the k-independent part (3.50), enables to write the generating function as

$$\xi_{[\alpha]}(j) = \mathrm{sdet}^{-1} J_r J_a \int d(Z, \tilde{Z}) \mathrm{sdet}(1 - Z\tilde{Z}) \int d\psi \; e^{-S[\Psi; Z, \tilde{Z}]}, \tag{3.63}$$

with

$$S[\Psi;Z,\tilde{Z}] = (\bar{\Psi}_{r1}\ \bar{\Psi}_{a1})\begin{pmatrix} 1 & \sqrt{S_\epsilon}Z\sqrt{S_\epsilon}^\dagger \\ \sqrt{S_\epsilon}^\dagger Z^\tau \sqrt{S_\epsilon} & 1 \end{pmatrix}\begin{pmatrix} \Psi_{r1} \\ \Psi_{a1} \end{pmatrix} + (\bar{\Psi}_{r2}\ \bar{\Psi}_{a2})\begin{pmatrix} \mathcal{J}_r^{-1} & \tilde{Z}^\tau \\ \tilde{Z} & \mathcal{J}_a^{-1} \end{pmatrix}\begin{pmatrix} \Psi_{r2} \\ \Psi_{a2} \end{pmatrix}. \tag{3.64}$$

The effect of the spectral average and of the color-flavor transformation is basically to replace (3.49) with (3.64). Notice also that both the matrices in (3.49) and both the matrices (3.64) are invariant under generalized transposition.

The integral over the supervector ψ, which is still gaussian, can now be performed. One gets

$$\xi_{[\alpha]}(j) = \mathrm{sdet}^{-1} J_r J_a \int d(Z,\tilde{Z})\ \mathrm{sdet}(1-Z\tilde{Z})\mathrm{sdet}^{-1/2}\begin{pmatrix} \mathcal{J}_r^{-1} & \tilde{Z}^\tau \\ \tilde{Z} & \mathcal{J}_a^{-1} \end{pmatrix} \mathrm{sdet}^{-1/2}\begin{pmatrix} 1 & \sqrt{S_\epsilon}Z\sqrt{S_\epsilon}^\dagger \\ \sqrt{S_\epsilon}^\dagger Z^\tau \sqrt{S_\epsilon} & 1 \end{pmatrix}. \tag{3.65}$$

Then, applying the formula for superdeterminants of blocks matrices in Appendix D yields the following result.

Theorem 3.4 *The generating function defined in (3.4) has the supersymmetric nonlinear σ model representation*

$$\xi_{[\alpha]}(j) = \int d(Z,\tilde{Z})\ sdet(1-Z\tilde{Z})\ sdet^{-1/2}(1-ZS_\epsilon^\dagger Z^\tau S_\epsilon)\ sdet^{-1/2}(1-\mathcal{J}_r\tilde{Z}^\tau\mathcal{J}_a\tilde{Z}).$$

The formula in the next corollary turns out to be the right expression to consider to perform a saddle-point analysis. It is directly found from the theorem by using the formula sdet = exp str log.

Corollary 3.5

$$\xi_{[\alpha]}(j) = \int d(Z,\tilde{Z})\ e^{-S[Z,\tilde{Z}]},$$

where the function $S[Z,\tilde{Z}]$ *is called* action, *or* exact action *in order to distinguish between* S *and its subsequent approximations, and is defined by*

$$\begin{aligned} S[Z,\tilde{Z}] &= -str\log(1-Z\tilde{Z}) + \frac{1}{2}str\log(1-Z\mathcal{S}_\epsilon^\dagger Z^\tau \mathcal{S}_\epsilon) \\ &\quad + \frac{1}{2}str\log(1-\mathcal{J}_r\tilde{Z}^\tau\mathcal{J}_a\tilde{Z}). \end{aligned}$$

Chapter 4

Mean Field Theory

4.1 The Zero Mode

4.1.1 Description

As the retarded and advanced sources are both set to zero, the exact action $S[Z, \tilde{Z}]$ in Corollary 3.5 becomes

$$\begin{aligned} S_0[Z,\tilde{Z}] &= -\mathrm{str}\log(1-Z\tilde{Z}) + \frac{1}{2}\mathrm{str}\log(1-ZS_\epsilon^\dagger Z^\intercal S_\epsilon) \\ &\quad + \frac{1}{2}\mathrm{str}\log(1-\tilde{Z}^\intercal\tilde{Z}). \end{aligned} \tag{4.1}$$

The goal here is to exhibit the subset of supermatrices $(Z_0, \tilde{Z}_0)$ around which the action S_0 is stationary. This particular subset of matrices is called *mean field mode*, or *zero mode*. We are thus interested in the equation

$$D_W S_0[Z_0, \tilde{Z}_0] = 0, \quad \text{for all} \quad W \in L(TR \otimes \mathcal{A}|TR \otimes \mathcal{A}). \tag{4.2}$$

Here, $D_W S_0$ stands for the variation of S_0 in the direction W, and its exact definition is given in Appendix E. The equation (4.2) is solved in the same appendix, and the mean field mode is found to consist in the set of matrices

$$\begin{aligned} &Z_0 = \mathbb{1}_{\mathcal{A}} \otimes Y \quad \text{and} \quad \tilde{Z}_0 = \mathbb{1}_{\mathcal{A}} \otimes \tilde{Y}, \\ &\text{with} \quad Y, \tilde{Y} \in L(TR|TR) \quad \text{such that} \quad \tilde{Y} = Y^\intercal. \end{aligned} \tag{4.3}$$

Moreover, Y and $\tilde{Y}$ must be diagonal in time-reversal space if time-reversal symmetry is broken. Of course, the color-flavor relations must still be satisfied, that is, the identities $\tilde{Y}_{\mathrm{BB}} = Y_{\mathrm{BB}}^{\dagger}$ and $\tilde{Y}_{\mathrm{FF}} = -Y_{\mathrm{FF}}^{\dagger}$ are fulfilled, and the eigenvalues of $Y_{\mathrm{BB}}^{\dagger}Y_{\mathrm{BB}}$ must have moduli smaller than one.

The set of all supermatrices Y satisfying the relations presented above parametrizes a supermanifold. Sometimes, other coordinates for this geometric object turn out to be more convenient to work with, and the transition between several useful sets of coordinates can be better understood in terms of Q matrices. In order to define these Q matrices, one starts by reintroducing the retarded-advanced space $RA = \mathbb{C}^2$ already used in the derivation of Theorem 3.4, and one defines the supermatrix R and its inverse by

$$R \equiv \begin{pmatrix} 1 & Y \\ \tilde{Y} & 1 \end{pmatrix}_{RA} \quad \text{and} \quad R^{-1} = \begin{pmatrix} \frac{1}{1-Y\tilde{Y}} & -Y\frac{1}{1-\tilde{Y}Y} \\ -\tilde{Y}\frac{1}{1-Y\tilde{Y}} & \frac{1}{1-\tilde{Y}Y} \end{pmatrix}_{RA}, \tag{4.4}$$

These supermatrices belong to $L(RA \otimes TR | RA \otimes TR)$. Then one sets

$$Q \equiv R\sigma_3^{RA}R^{-1}, \tag{4.5}$$

where σ_3^{RA} stands for the third Pauli matrix in retarded-advanced space. These supermatrices are precisely the object of Appendix F. It is explained in the section F.1 that they span a Lie supergroup, the so-called Efetov's σ model space. However, the main point here is to relate the supermatrices Y and $\tilde{Y}$ to the components of Q in retarded-advanced space. It is not difficult to obtain the following correspondences,

$$Q_{rr} - 1 = 2Y\tilde{Y}(1 - Y\tilde{Y})^{-1} \quad , \quad Q_{rr} + 1 = 2(1 - Y\tilde{Y})^{-1}, \tag{4.6}$$

$$Q_{aa} + 1 = -2\tilde{Y}Y(1 - \tilde{Y}Y)^{-1} \quad , \quad Q_{aa} - 1 = -2(1 - \tilde{Y}Y)^{-1}, \tag{4.7}$$

$$Q_{ra} = -2Y(1 - \tilde{Y}Y)^{-1}, \tag{4.8}$$

$$Q_{ar} = 2\tilde{Y}(1 - Y\tilde{Y})^{-1}. \tag{4.9}$$

In Appendix F.2, the Q matrices are parametrized by means of polar coordinates. These coordinates are the ones that will subsequently be used to

perform the explicit calculations. The definitions of these coordinates will not be written here since they involve cumbersome expressions. The reader interested in carrying out the whole of the forthcoming calculations by himself can refer to this appendix.

4.1.2 The Mean Field Generating Function

Let us now turn back to the generating function $\xi_{[\alpha]}(j)$ introduced in Chapter 3, and let us define the *mean field generating function* $\xi^{\rm MF}_{[\alpha]}(j)$ by the formula in Theorem 3.4, or equivalently in Corollary 3.5, where the integrals are restricted onto the zero mode $(Z_0, \tilde{Z}_0)$. We will merely write dQ for the corresponding measure when the integrand is expressed in terms of Q matrices. With these definitions, the mean field generating function can be expressed as

$$\xi^{\rm MF}_{[\alpha]}(j) = \int dQ\, e^{-S_0^{\rm MF}} P_{[\alpha]}(j), \tag{4.10}$$

where $S_0^{\rm MF}$ is the source-free action (4.1) restricted onto the zero mode and $P_{[\alpha]}(j)$ is a supersymmetry breaking factor. In the sequel, the Q matrices and the sources $j^T = (j_a, j_r{}^T)$ will always enter the formulae through the combinations

$$\hat{Q} \equiv \sigma_3^{RA} Q - \mathbb{1} \quad \text{and} \quad M_{[\alpha]}(j) \equiv \begin{pmatrix} j_r \cdot \mathcal{E}^{(r)} & 0 \\ 0 & j_a \mathcal{E}^{(a)} \end{pmatrix}_{RA}. \tag{4.11}$$

In particular, the Bose-Bose block of the supermatrix $\hat{Q}$ will play a particularly important role, and we shall use the simpler notation

$$\hat{Q}^B \equiv \hat{Q}_{BB}. \tag{4.12}$$

The starting point of all the forthcoming mean field calculations is the two formulae given in the next theorem.

Theorem 4.1 *In the linear approximation in ϵ, the mean field source-free action S_0^{MF} reads*

$$S_0^{MF} = \frac{B\epsilon}{2} str\, \hat{Q}.$$

Moreover, a possible supersymmetry breaking factor $P_\beta(j)$ satisfying the identity (4.10) is given by

$$P_{[\alpha]}(j) = \det\left[\mathbb{1} - \frac{1}{2}M_{[\alpha]}(j)\hat{Q}^B\right]^{-\frac{1}{2}}.$$

Proof. It is straightforward to see that the exact action S as written in Corollary 3.5 restricted onto the zero mode reads

$$\begin{aligned} S^{\mathrm{MF}} &= -\mathrm{str}\log\mathbb{1}_{\mathcal{A}}\otimes\left(1-Y\tilde{Y}\right)+\frac{1}{2}\mathrm{str}\log\mathbb{1}_{\mathcal{A}}\otimes\left(1-e^{-2\epsilon}Y\tilde{Y}\right) \\ &\quad +\frac{1}{2}\mathrm{str}\log\left(1-\mathcal{J}_r Y\mathcal{J}_a\tilde{Y}\right). \end{aligned} \tag{4.13}$$

As the sources j_r and j_a are set to zero in (4.13), or equivalently as the source-free action (4.1) is restricted onto the zero mode, one finds

$$\begin{aligned} S_0^{\mathrm{MF}} &= \frac{2B}{2}\mathrm{str}\left[\log\left(1-e^{-2\epsilon}Y\tilde{Y}\right)-\log\left(1-Y\tilde{Y}\right)\right] \\ &= 2B\epsilon\ \mathrm{str}\frac{Y\tilde{Y}}{1-Y\tilde{Y}}+\mathcal{O}(\epsilon^2). \end{aligned} \tag{4.14}$$

The supermatrix involved in the last expression can directly be written with the retarded-retarded component (4.6) of the Q matrix. To get the expression as in the theorem, one first uses the invariance of the supertrace over cyclic permutations and writes

$$\begin{aligned} S_0^{\mathrm{MF}} &= B\epsilon\ \mathrm{str}\left(\frac{Y\tilde{Y}}{1-Y\tilde{Y}}+\frac{\tilde{Y}Y}{1-\tilde{Y}Y}\right)+\mathcal{O}(\epsilon^2) \\ &= \frac{B\epsilon}{2}\mathrm{str}\ \sigma_3^{RA}Q+\mathcal{O}(\epsilon^2). \end{aligned} \tag{4.15}$$

The last equality is easily obtained from the formulae (4.6) and (4.7) for the components of the Q matrix in retarded-advanced space. Let us now turn to the remaining part of the mean field action, that is the term containing the sources. We have here to use the identity $\mathrm{str}\,(\log A+\log B)=\mathrm{str}\log AB$, which holds for any supermatrices A and B so that these operations make sense. Then, from (4.13) and (4.14),

$$S^{\mathrm{MF}}-S_0^{\mathrm{MF}}=\frac{1}{2}\ \mathrm{str}\log\frac{1-\mathcal{J}_r Y\mathcal{J}_a\tilde{Y}}{1-Y\tilde{Y}}. \tag{4.16}$$

By Corollary 3.5, and by definition of $\xi^{\mathrm{MF}}_{[\alpha]}$, one has

$$\xi^{\mathrm{MF}}_{[\alpha]}(j) = \int dQ\, e^{-S_0^{\mathrm{MF}}} P_{[\alpha]}(j), \quad \text{where} \quad P_{[\alpha]}(j) = e^{-(S^{\mathrm{MF}} - S_0^{\mathrm{MF}})}. \tag{4.17}$$

The formula (4.16) enables us to express $P_{[\alpha]}$ in terms of Q matrices. Indeed,

$$\begin{aligned} P_{[\alpha]}(j) &= \frac{\mathrm{sdet}^{-\frac{1}{2}}\left(1 - \mathcal{J}_r Y \mathcal{J}_a \tilde{Y}\right)}{\mathrm{sdet}^{-\frac{1}{2}}\left(1 - Y\tilde{Y}\right)} \\ &= \mathrm{sdet}^{-\frac{1}{2}} \begin{pmatrix} 1 & \mathcal{J}_r Y \\ \mathcal{J}_a \tilde{Y} & 1 \end{pmatrix} \begin{pmatrix} 1 & Y \\ \tilde{Y} & 1 \end{pmatrix}^{-1}, \end{aligned} \tag{4.18}$$

where the retarded-advanced space has been reintroduced in the last expression. Making use of the explicit formula (4.4) for R^{-1} and performing the matrix product yields

$$P_{[\alpha]}(j) = \mathrm{sdet}^{-\frac{1}{2}} \begin{pmatrix} \left(1 - \mathcal{J}_r Y \tilde{Y}\right) \frac{1}{1 - Y\tilde{Y}} & (\mathcal{J}_r - 1) Y \frac{1}{1 - \tilde{Y}Y} \\ (\mathcal{J}_a - 1) \tilde{Y} \frac{1}{1 - Y\tilde{Y}} & \left(1 - \mathcal{J}_a \tilde{Y} Y\right) \frac{1}{1 - \tilde{Y}Y} \end{pmatrix}. \tag{4.19}$$

Then, the relations (4.6)-(4.9) relating Y and $\tilde{Y}$ to the components of Q in retarded-advanced space lead to the new expression

$$\mathrm{sdet}^{-\frac{1}{2}} \left[\mathbb{1} + \frac{1}{2} \begin{pmatrix} 1 - \mathcal{J}_r & 0 \\ 0 & 1 - \mathcal{J}_a \end{pmatrix} \begin{pmatrix} Q_{rr} - 1 & Q_{ra} \\ -Q_{ar} & -Q_{aa} - 1 \end{pmatrix} \right] \tag{4.20}$$

for $P_{[\alpha]}$. Moreover, the definitions (3.1) and (3.2) of the source supermatrices J_r and J_a imply that

$$P_{[\alpha]}(j) = \mathrm{sdet}^{-\frac{1}{2}} \left[\mathbb{1} - \frac{E_B}{2} \begin{pmatrix} j_r \cdot \mathcal{E}^{(r)} & 0 \\ 0 & j_a \mathcal{E}^{(a)} \end{pmatrix}_{ItA} (\sigma_3^{ItA} Q - 1) \right]. \tag{4.21}$$

Finally, the formula for $P_{[\alpha]}$ given in the theorem follows from the definition of a superdeterminant, and from the definitions in (4.11).

□

Property 4.2 *For all j_a and $j_r = (j_1, \ldots, j_{q-1})$ in a sufficiently small neighborhood of the origin*

$$\xi^{MF}_{[\alpha]}(j_a, 0) = \xi^{MF}_{[\alpha]}(0, j_r) = 1.$$

This property states that Property 3.2 remains fulfilled at the mean field level. The proof can be found in Appendix F.4.

4.2 The Mean Field Autocorrelation Functions

4.2.1 Principles

The mean field generating function provides formulae for the statistical quantities one wants to compute. Indeed, one can define the *mean field autocorrelation functions* $C^{\mathrm{MF}}_{[\alpha]}$ by substituting the mean field generating function for the exact one in Theorem 3.1. In other words,

$$C^{\mathrm{MF}}_{[\alpha]} \equiv \lim_{\epsilon \to 0} \frac{(2\epsilon)^{q-1}}{2B(q-1)!} \delta \xi^{\mathrm{MF}}_{[\alpha]}. \tag{4.22}$$

The calculation of these mean field quantities is precisely the object of this section. Let us just make here some general remarks about the procedure that will be used. The derivatives δ defined in (3.11) and the superintegral in (4.10) are commuted. This leads to the formula

$$C^{\mathrm{MF}}_{[\alpha]} = \lim_{\epsilon \to 0} \frac{(2\epsilon)^{q-1}}{2B(q-1)!} \int dQ \; e^{-S_0^{\mathrm{MF}}} \delta P_{[\alpha]}, \tag{4.23}$$

where $\delta P_{[\alpha]}$ denotes the derivatives

$$\delta P_{[\alpha]} \equiv \prod_{s=0}^{q-1} \frac{\partial}{\partial j_s} P_{[\alpha]}(\mathbf{0}). \tag{4.24}$$

This formula implies that the determinant defining the supersymmetry breaking factor $P_{[\alpha]}$ in Theorem 4.1 has to be differentiated q times. First, for any $\alpha \in \mathbb{R}$ and any permutation σ of $q \in \mathbb{N}$ elements, one writes

$$\rho_\alpha(\sigma) \equiv \alpha^{\text{number of cycles in } \sigma}. \tag{4.25}$$

Then, by Theorem C.1,

$$\begin{aligned} \delta P_{[\alpha]} \;=\; & \frac{1}{2^q} \sum_{\sigma \in S_q} \rho_{\frac{1}{2}}(\sigma) \sum_{\substack{x_0,\dots,x_{q-1}=r,a \\ \gamma_0,\dots,\gamma_{q-1} \in \mathbb{N}_{2B} \\ t_0,\dots t_{q-1}=\uparrow,\downarrow}} \left[E^a \mathcal{E}^{\alpha_0,\alpha_0} \hat{Q}_{BB} \right]^{x_0, x_{\sigma(0)}}_{\gamma_0, \gamma_{\sigma(0)}, t_0, t_{\sigma(0)}} \\ & \cdot \prod_{j=1}^{q-1} \left[E^r \mathcal{E}^{\alpha_j,\alpha_j} \hat{Q}_{BB} \right]^{x_j, x_{\sigma(j)}}_{\gamma_j, \gamma_{\sigma(j)}, t_j, t_{\sigma(j)}} \end{aligned} \tag{4.26}$$

Notice that the permutations $\sigma \in S_q$ act here on the set $\{0, \ldots, q-1\}$. This formula can be seen as a sum over all possible configurations of the type (x, γ, t, σ) where $x \in \{r, a\}^q$, $\gamma \in \mathbb{N}_{2B}{}^q$ and $t \in \{\uparrow, \downarrow\}^q$. However, most of these configurations do not contribute to $\delta P_{[\alpha]}$. For example, the projectors E^r and E^a imply that a configuration (x, γ, t, σ) does not contribute unless $x_0 = a$ and $x_j = r$ for all $j \in \mathbb{N}_{q-1}$. Consequently, $\delta P_{[\alpha]}$ can actually be seen as a sum over configurations of the simpler type (γ, t, σ). Furthermore, only a few choices for γ yield non-vanishing contributions, and the set of these possible choices depends on the list of directed bonds $[\alpha]$ in (2.56), that is on the autocorrelation function one wants to compute. In fact, in a term (γ, t, σ) of (4.26), the matrices $\mathcal{E}^{\alpha_j, \alpha_j}$ are the only objects influenced by γ. They are however diagonal in time-reversal space and have no structure in retarded-advanced space. One can thus write (4.26) as

$$\delta P_{[\alpha]} = \frac{1}{2^q} \sum_{\sigma \in S_q} \rho_{\frac{1}{2}}(\sigma) \sum_{t \in \{\uparrow,\downarrow\}^q} F_{[\alpha]}(t, \sigma) \pi(t, \sigma), \tag{4.27}$$

where

$$F_{[\alpha]}(t, \sigma) \equiv \sum_{\gamma \in (\mathbb{N}_{2B})^q} \prod_{j=0}^{q-1} \left[\mathcal{E}^{\alpha_j, \alpha_j} \right]_{t_j, t_j}^{\gamma_j, \gamma_{\sigma(j)}} \tag{4.28}$$

is the $[\alpha]$-dependent factor, and

$$\pi(t, \sigma) \equiv \begin{cases} \hat{Q}^B_{aa;t_0,t_0} \prod_{j=1}^{q-1} \hat{Q}^B_{rr;t_j,t_{\sigma(j)}} & \text{if } \sigma(0) = 0 \\ \hat{Q}^B_{ar;t_0,t_{\sigma(0)}} \hat{Q}^B_{ra;t_i,t_0} \prod_{\substack{j=1 \\ j \neq i}}^{q-1} \hat{Q}^B_{rr;t_j,t_{\sigma(j)}} & \text{if } \sigma(i) = 0, i \in \mathbb{N}_{q-1} \end{cases} \tag{4.29}$$

contains the product of Q matrices. The definition of $\pi(t, \sigma)$ shows that, in principle, we should distinguish between the permutations σ in (4.27) which leave the element 0 unchanged and those moving this element. This point will be clarified in the sequel. The formula (4.28) can easily be simplified. Indeed, with the notations

$$\uparrow [\beta] \equiv \beta \quad \text{and} \quad \downarrow [\beta] = \hat{\beta}, \tag{4.30}$$

the factor $F_{[\alpha]}(t,\sigma)$ becomes

$$\begin{aligned} F_{[\alpha]}(t,\sigma) &= \sum_{\gamma\in(\mathbb{N}_{2B})^q}\prod_{j=0}^{q-1}\delta_{\gamma_j,t_j[\alpha_j]}\delta_{\gamma_{\sigma(j)},t_j[\alpha_j]} \\ &= \sum_{\gamma\in(\mathbb{N}_{2B})^q}\prod_{j=0}^{q-1}\delta_{\gamma_j,t_j[\alpha_j]}\delta_{\gamma_j,t_{\sigma^{-1}(j)}[\alpha_{\sigma^{-1}(j)}]} \\ &= \prod_{j=0}^{q-1}\delta_{t_j[\alpha_j],t_{\sigma(j)}[\alpha_{\sigma(j)}]}. \end{aligned} \tag{4.31}$$

Hence, $F_{[\alpha]}(t,\sigma)$ is either equal to 1 or to 0, and any choice for the list of directed bonds $[\alpha]$ selects a corresponding set of contributing configurations (t,σ).

The expression (4.27) for $\delta P_{[\alpha]}$ enables to write the mean field autocorrelation functions $C^{\mathrm{MF}}_{[\alpha]}$ given in (4.23) as

$$C^{\mathrm{MF}}_{[\alpha]} = \sum_{\sigma\in S_q}\rho_{\frac{1}{2}}(\sigma)\sum_{t\in\{\uparrow,\downarrow\}^q}F_{[\alpha]}(t,\sigma)I_\pi(t,\sigma), \tag{4.32}$$

where, for any $t=(t_0,\ldots,t_{q-1})$ in $\{\uparrow,\downarrow\}^q$, and for any $\sigma\in S_q$,

$$I_\pi(t,\sigma) \equiv \lim_{\epsilon\to 0}\frac{\epsilon^{q-1}}{4B(q-1)!}\int dQ\; e^{-S_0^{\mathrm{MF}}}\pi(t,\sigma). \tag{4.33}$$

This last expression is a mean field integral in the small ϵ regime. These integrals are precisely the object of Appendix F.3, and their values depend on the symmetry class.

4.2.2 Unitary Symmetry

The goal of this paragraph is to compute the mean field integrals (4.33) in the unitary symmetry class. For this purpose, the components of Q matrices are expressed in terms of the polar coordinates described in Appendix F.2.2. They are written

$$\begin{aligned} \hat{Q}^B_{rr} &= u_{1BB}u_{2B}\cosh\theta_B\bar{u}_{2B}\bar{u}_{1BB} + u_{1BF}u_{2F}\cos\theta_F\bar{u}_{2F}\bar{u}_{1FB} - 1 \\ \hat{Q}^B_{aa} &= v_{1BB}v_{2B}\cosh\theta_B\bar{v}_{2B}\bar{v}_{1BB} + v_{1BF}v_{2F}\cos\theta_F\bar{v}_{2F}\bar{v}_{1FB} - 1 \\ \hat{Q}^B_{ra} &= -u_{1BB}u_{2B}\sinh\theta_B\bar{v}_{2B}\bar{v}_{1BB} + u_{1BF}u_{2F}i\sin\theta_F\bar{v}_{2F}\bar{v}_{1FB} \\ \hat{Q}^B_{ar} &= -v_{1BB}v_{2B}\sinh\theta_B\bar{u}_{2B}\bar{u}_{1BB} + v_{1BF}v_{2F}i\sin\theta_F\bar{u}_{2F}\bar{u}_{1FB}. \end{aligned} \tag{4.34}$$

The supermatrices u_1, $\bar{u}_1$, v_1 and $\bar{v}_1$ involve 4 anticommuting variables which are contained in two matrices,

$$\eta = \begin{pmatrix} \eta_1 & 0 \\ 0 & -\eta_1^* \end{pmatrix}_{TR} \quad \text{and} \quad \kappa = \begin{pmatrix} \kappa_1 & 0 \\ 0 & -\kappa_1^* \end{pmatrix}_{TR}. \tag{4.35}$$

The even combinations of Grassmann variables

$$|\eta| \equiv 1 - 2\eta_1^*\eta_1 \quad \text{and} \quad |\kappa| \equiv 1 + 2\kappa_1^*\kappa_1 \tag{4.36}$$

are of particular importance. Making use of the formulae in Appendix F.2.2, the decomposition (4.34) can be made explicit. It reads

$$\hat{Q}^B_{rr} = \left(|\eta|^2 \cosh\theta_1 + 4\eta_1^*\eta_1 \cos\theta - 1\right) \cdot \mathbb{1}_{TR} \tag{4.37}$$

$$\hat{Q}^B_{aa} = \left(|\kappa|^2 \cosh\theta_1 - 4\kappa_1^*\kappa_1 \cos\theta - 1\right) \cdot \mathbb{1}_{TR} \tag{4.38}$$

$$\hat{Q}^B_{ra} = -|\eta||\kappa| e^{i\xi\sigma_3^{TR}} \sinh\theta_1 - 4\eta^\dagger\kappa e^{i\phi\sigma_3^{TR}} \sin\theta \tag{4.39}$$

$$\hat{Q}^B_{ar} = -|\eta||\kappa| e^{-i\xi\sigma_3^{TR}} \sinh\theta_1 - 4\kappa^\dagger\eta e^{-i\phi\sigma_3^{TR}} \sin\theta \tag{4.40}$$

Here, $\theta_1 \in \mathbb{R}_+$, $\theta \in [0, \pi)$, and $\xi, \phi \in [0, 2\pi)$ denote the four real coordinates.

In Appendix F.3, it is found that, among all the terms in (4.37)-(4.40), only the leading ones as $\lambda_1 \equiv \cosh\theta_1 \to \infty$ are relevant and determine the value of (4.33). Two expressions having the same leading terms are said to be equivalent and this relation is denoted by $\overset{2}{\sim}$, as in (F.97). For example,

$$\sinh\theta_1 = \sqrt{\cosh^2\theta_1 - 1} \overset{2}{\sim} \cosh\theta_1 \equiv \lambda_1. \tag{4.41}$$

Therefore, the components (4.37)-(4.40) of $\hat{Q}^B$ can be considerably simplified, and one can consider instead

$$\hat{Q}^B \overset{2}{\sim} U_{2B} \cdot x \cdot U_{2B}^\dagger, \quad x \equiv \lambda_1 \cdot \begin{pmatrix} |\eta|^2 & -|\eta||\kappa| \\ -|\eta||\kappa| & |\kappa|^2 \end{pmatrix}_{RA}, \tag{4.42}$$

with

$$U_{2B} \equiv \begin{pmatrix} u_{2B} & 0 \\ 0 & v_{2B} \end{pmatrix}_{RA} \equiv \begin{pmatrix} e^{i\xi\sigma_3^{TR}} & 0 \\ 0 & \mathbb{1}_{TR} \end{pmatrix}_{RA} \tag{4.43}$$

Let us now express the product $\pi(t,\sigma)$ defined in (4.29) by means of the polar coordinates. Making use of (4.29), (4.42) and (4.43), it can be readily checked that

$$\pi(t,\sigma) \overset{2}{\sim} \prod_{j=0}^{q-1} \delta_{t_j,t_{\sigma(j)}} \begin{cases} x_{aa}x_{rr}^{q-1} & \text{if } \sigma(0)=0 \\ x_{ar}x_{ra}x_{rr}^{q-2} & \text{if } \sigma(i)=0, i\in\mathbb{N}_{q-1} \end{cases} \tag{4.44}$$

$$\overset{2}{\sim} |\eta|^{2(q-1)}|\kappa|^2 \cdot \lambda_1^q \cdot \prod_{j=0}^{q-1} \delta_{t_j,t_{\sigma(j)}}. \tag{4.45}$$

As a consequence, in the unitary mean field integrals (4.33), there is no need to distinguish between the permutations leaving the element 0 unchanged and those moving this element.

It is also explained in Appendix F.3 that the terms of $\pi(t,\sigma)$ which do not involve all the four anticommuting variables cannot contribute to the mean field integral $I_\pi(t,\sigma)$. Retaining from (4.45) only the monomial of highest degree in the anticommuting variables yields

$$\pi(t,\sigma) \sim -2^4(q-1)\eta_1^*\eta_1\kappa_1^*\kappa_1 \cdot \lambda_1^q \cdot \prod_{j=0}^{q-1} \delta_{t_j,t_{\sigma(j)}}. \tag{4.46}$$

Here, the symbol $\sim$ stands for the equivalence relation defined in (F.100). Once this formula is plugged into the mean field integral (4.33), one finds

$$I_\pi(t,\sigma) \equiv \lim_{\epsilon\to 0} \epsilon^{q-1} \int dQ\ e^{-S_0^{\mathrm{MF}}} f_q(Q). \tag{4.47}$$

with

$$f_q(Q) \equiv \frac{-2^3}{2B(q-2)!} \cdot \lambda_1^q \cdot \eta_1^*\eta_1\kappa_1^*\kappa_1 \cdot \prod_{j=0}^{q-1} \delta_{t_j,t_{\sigma(j)}}. \tag{4.48}$$

The expression (4.47) for $I_\pi(t,\sigma)$ is exactly of the form (F.88) with $n=q$. Indeed, $f_q(Q)$ in (4.48) has the wanted properties (F.86) and (F.87). By the identity (F.88), the result reads

$$I_\pi(t,\sigma) = \frac{1}{(2B)^q} \prod_{j=0}^{q-1} \delta_{t_j,t_{\sigma(j)}}. \tag{4.49}$$

Formally, the use of the identity (F.88) requires to check beforehand that the term of $\pi(t,\sigma)$ containing the 4 Grassmann variables contains a factor killing

the discontinuity $(\lambda_1 - \lambda)^{-2}$ in the measure dQ. This can easily be done using the exact formulae (4.37)-(4.40). However, this short calculation is not reproduced here.

4.2.3 Orthogonal Symmetry

We now have to compute the mean field integrals (4.33) in the orthogonal symmetry class. The components of $\hat{Q}^B$ are expressed in terms of the polar coordinates described in (F.2.3). As in the unitary case, these components are decomposed as

$$\begin{aligned}
\hat{Q}^B_{rr} &= u_{1BB}u_{2B}\cosh\theta_B\bar{u}_{2B}\bar{u}_{1BB} + u_{1BF}u_{2F}\cos\theta_F\bar{u}_{2F}\bar{u}_{1FB} - 1 \\
\hat{Q}^B_{aa} &= v_{1BB}v_{2B}\cosh\theta_B\bar{v}_{2B}\bar{v}_{1BB} + v_{1BF}v_{2F}\cos\theta_F\bar{v}_{2F}\bar{v}_{1FB} - 1 \\
\hat{Q}^B_{ra} &= -u_{1BB}u_{2B}\sinh\theta_B\bar{v}_{2B}\bar{v}_{1BB} + u_{1BF}u_{2F}i\sin\theta_F\bar{v}_{2F}\bar{v}_{1FB} \\
\hat{Q}^B_{ar} &= -v_{1BB}v_{2B}\sinh\theta_B\bar{u}_{2B}\bar{u}_{1BB} + v_{1BF}v_{2F}i\sin\theta_F\bar{u}_{2F}\bar{u}_{1FB}. \qquad (4.50)
\end{aligned}$$

The supermatrices u_1, $\bar{u}_1$, v_1 and $\bar{v}_1$ involve 8 anticommuting variables, which are contained in the two matrices

$$\eta \equiv \begin{pmatrix} \eta_1^* & \eta_2 \\ \eta_2^* & \eta_1 \end{pmatrix}_{TR} \quad \text{and} \quad \kappa \equiv \begin{pmatrix} \kappa_1^* & \kappa_2 \\ \kappa_2^* & \kappa_1 \end{pmatrix}_{TR}. \qquad (4.51)$$

The even combinations

$$|\eta| \equiv 1 - 2(\eta_1^*\eta_1 + \eta_2^*\eta_2) + 12\eta_1^*\eta_1\eta_2^*\eta_2, \qquad (4.52)$$

$$|\kappa| \equiv 1 + 2(\kappa_1^*\kappa_1 + \kappa_2^*\kappa_2) + 12\kappa_1^*\kappa_1\kappa_2^*\kappa_2 \qquad (4.53)$$

of anticommuting variables will be of particular importance. In (4.50), the matrices $\cosh\theta_B$ and $\sinh\theta_B$ are

$$\cosh\theta_B = \begin{pmatrix} \cosh\theta_1\cosh\theta_2 & \sinh\theta_1\sinh\theta_2 \\ \sinh\theta_1\sinh\theta_2 & \cosh\theta_1\cosh\theta_2 \end{pmatrix}_{TR} \qquad (4.54)$$

and

$$\sinh\theta_B = \begin{pmatrix} \sinh\theta_1\cosh\theta_2 & \cosh\theta_1\sinh\theta_2 \\ \cosh\theta_1\sinh\theta_2 & \sinh\theta_1\cosh\theta_2 \end{pmatrix}_{TR}, \qquad (4.55)$$

with $\theta_1, \theta_2 \in \mathbb{R}_+$, whereas $\cos\theta_F$ and $\sin\theta_F$ follow from $\theta_F = \theta \cdot \mathbb{1}_{TR}$, where $\theta \in [0, \pi]$. Five other real parameters are contained in the supermatrices u_2, $\bar{u}_2$, v_2 and $\bar{v}_2$. Two of them, $\xi, \chi \in [0, 2\pi]$, span the set of matrices

$$U_{2B} \equiv \begin{pmatrix} u_{2B} & 0 \\ 0 & v_{2B} \end{pmatrix}_{RA} \equiv \begin{pmatrix} e^{i\xi\sigma_3^{TR}} & 0 \\ 0 & e^{i\chi\sigma_3^{TR}} \end{pmatrix}_{RA} \tag{4.56}$$

Similarly to the unitary symmetry case, the relevant region in the small ϵ regime of the orthogonal mean field integrals is where $\lambda_1 \equiv \cosh\theta_1$ and $\lambda_2 \equiv \cosh\theta_2$ are both asymptotically large. Two functions of Q that have the same asymptotics are said to be equivalent, and this relation is denoted by the symbol $\overset{2}{\sim}$ as in the unitary symmetry case. This property implies in particular that, in the right-hand sides of (4.50), it is sufficient to retain the first term. Moreover,

$$\cosh\theta_B \overset{2}{\sim} \sinh\theta_B \overset{2}{\sim} \lambda_1\lambda_2 \cdot \mathrm{one}(TR), \quad \mathrm{one}(TR) \equiv \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}_{TR}. \tag{4.57}$$

A short calculation shows that

$$\hat{Q}^B \overset{2}{\sim} U_{2B} \cdot x \otimes \mathrm{one}(TR) \cdot U_{2B}^{\dagger}, \quad x \equiv \lambda_1\lambda_2 \cdot \begin{pmatrix} |\eta|^2 & -|\eta||\kappa| \\ -|\eta||\kappa| & |\kappa|^2 \end{pmatrix}_{RA} \tag{4.58}$$

which is the orthogonal analog of (4.42). The relations (4.58) enable to express the function $\pi(\boldsymbol{t}, \sigma)$ defined in (4.29) in a very convenient way. Notice first that, for any $\psi, \psi' \in [0, 2\pi]$, and for any $t, t' \in \{\uparrow, \downarrow\}$,

$$N(\psi, \psi')_{t,t'} \equiv \left[e^{i\psi\sigma_3^{TR}} \cdot \mathrm{one}(TR) \cdot e^{-i\psi'\sigma_3^{TR}}\right]_{t,t'} = e^{i[s(t)\psi - s(t')\psi']}. \tag{4.59}$$

Suppose now that $\sigma \in S_q$ is such that $\sigma(0) = 0$. Then, it is not difficult to see that (4.29) yields

$$\pi(\boldsymbol{t}, \sigma) \overset{2}{\sim} x_{aa}x_{rr}^{q-1}N(\chi, \chi)_{t_0,t_0}\prod_{j=1}^{q-1} N(\xi, \xi)_{t_j,t_{\sigma(j)}} = x_{aa}x_{rr}^{q-1}. \tag{4.60}$$

On the other hand, if $\sigma(i) = 0$ for some $i \in \mathbb{N}_q$, then

$$\begin{aligned} \pi(\boldsymbol{t}, \sigma) &\overset{2}{\sim} x_{ar}x_{ra}x_{rr}^{q-2}N(\chi, \xi)_{t_0,t_{\sigma(0)}}N(\xi, \chi)_{t_i,t_0}\prod_{\substack{j=1 \\ j\neq i}}^{q-1} N(\xi, \xi)_{t_j,t_{\sigma(j)}} \\ &\overset{2}{\sim} x_{ar}x_{ra}x_{rr}^{q-2}. \end{aligned} \tag{4.61}$$

But (4.58) shows that the right-hand sides of (4.60) and (4.61) are identical, and hence, there is no need to treat the permutations leaving 0 unchanged separately from the others. The situation is thus completely similar to the one we got in (4.45) concerning the unitary symmetry class. Here, one merely have

$$\pi(t,\sigma) \overset{2}{\sim} |\eta|^{2(q-1)}|\kappa|^2 \cdot (\lambda_1\lambda_2)^q \tag{4.62}$$

for all permutations $\sigma \in S_q$. Nonetheless this result is independent of σ, but it is also independent of the time-reversal configuration $t \in \{\uparrow,\downarrow\}^q$. The mean field integral $I_\pi(t,\sigma)$ in (4.33) can then be performed, or the result can also directly be inferred from the unitary result (4.49) and the arguments presented in 2.2.2. In any case, one finds

$$I_\pi(t,\sigma) = \frac{1}{(2B)^q}. \tag{4.63}$$

4.2.4 Explicit Formulae

From (4.31), (4.32), (4.49) and (4.63), the mean field autocorrelation functions take the form

$$C^{\mathrm{MF}}_{[\alpha]} = \frac{1}{(2B)^q} \sum_{\substack{\sigma\in S_q \\ t\in\{\uparrow,\downarrow\}^q}} \rho_{\frac{1}{2}}(\sigma) \prod_{j=0}^{q-1} \delta_{t_j[\alpha_j],t_{\sigma(j)}[\alpha_{\sigma(j)}]} \begin{cases} \delta_{t_j,t_{\sigma(j)}} & \text{(U)} \\ 1 & \text{(O)} \end{cases} \tag{4.64}$$

The first line stands for the unitary symmetry class, whereas the second line stands for the orthogonal symmetry class. This formula can also be written

$$C^{\mathrm{MF}}_{[\alpha]} = \frac{1}{(2B)^q} \sum_{\substack{\sigma\in S_q \\ t\in\{\uparrow,\downarrow\}^q}} \rho_{\frac{1}{2}}(\sigma) \prod_{j=0}^{q-1} \begin{cases} \delta_{t_j,t_{\sigma(j)}}\delta_{\alpha_j,\alpha_{\sigma(j)}} & \text{(U)} \\ \delta_{t_j[\alpha_j],t_{\sigma(j)}[\alpha_{\sigma(j)}]} & \text{(O)} \end{cases} \tag{4.65}$$

This is the most general formula for the mean field autocorrelation functions. It can now be specialized to several particular cases.

The Moments

The mean field moments read

$$M^{\mathrm{MF}}_\beta(q) \equiv \left\langle |a_\beta|^{2q} \right\rangle^{\mathrm{MF}}. \tag{4.66}$$

The corresponding list of directed bonds $[\alpha]$ introduced in (2.56) is

$$\alpha_j \equiv \beta, \quad \text{for all } 0 \leq j \leq q-1. \tag{4.67}$$

With this choice, the general formula (4.65) becomes

$$M_\beta^{\mathrm{MF}}(q) = \frac{1}{(2B)^q} \sum_{\substack{\sigma \in S_q \\ t \in \{\uparrow,\downarrow\}^q}} \rho_{\frac{1}{2}}(\sigma) \prod_{j=0}^{q-1} \delta_{t_j, t_{\sigma(j)}} \tag{4.68}$$

independently of the symmetry class.

In (4.68), the product of Kronecker symbols imposes to the permutations σ to commute the integers j for which $t_j = \uparrow$ together. Let us consider a configuration $\boldsymbol{t} \in \{\uparrow,\downarrow\}^q$ and let $n \in \{0,\dots,q\}$ be the number of indices j with $t_j = \uparrow$. The configuration $\boldsymbol{t}$ is characterized by the two increasing functions

$$f_\uparrow : \mathbb{N}_n \to \{0,\dots,q-1\} \quad \text{and} \quad f_\downarrow : \mathbb{N}_{q-n} \to \{0,\dots,q-1\} \tag{4.69}$$

satisfying the condition

$$t_j = \begin{cases} \uparrow & \text{if } j \in \mathrm{Range}(f_\uparrow) \\ \downarrow & \text{if } j \in \mathrm{Range}(f_\downarrow) \end{cases} \tag{4.70}$$

In fact, $f_\downarrow$ is completely determined by $f_\uparrow$. Then, the formula

$$f(j) \equiv \begin{cases} f_\uparrow(j) & \text{if } 1 \leq j \leq n \\ f_\downarrow(j-n) & \text{if } n < j \leq q \end{cases} \tag{4.71}$$

defines a bijection from $\mathbb{N}_q$ to $\{0,\dots,q-1\}$. The set of such bijections is denoted by $\mathcal{F}_{n,q-n}$ and the disjoint union of these sets, for $0 \leq n \leq q$, can be seen as the set $\{\uparrow,\downarrow\}^q$ of all the possible time-reversal configurations $\boldsymbol{t}$. Then, for some time-reversal configuration $f \in \mathcal{F}_{n,q-n}$, the permutations $\sigma \in S_q$ that contribute to the mean field moments (4.68) are the ones fulfilling the condition

$$\sigma\big(f(\mathbb{N}_n)\big) = f(\mathbb{N}_n). \tag{4.72}$$

The set of such permutations is denoted by $S_{n,q-n}[f]$, and we simply write $S_{n,q-n}$ for $S_{n,q-n}[\mathrm{id}]$. The formula (4.72) shows that, for all $f \in \mathcal{F}_{n,q-n}$,

$$S_{n,q-n}[f] = f S_{n,q-n} f^{-1}. \tag{4.73}$$

Moreover, it is easy to convince oneself that this conjugation leaves the cyclic structure invariant. In particular, the ρ factor is not affected by this conjugation. Putting al these definitions and remarks together, the mean field moments (4.68) become

$$\begin{aligned} M_\beta^{\mathrm{MF}}(q) &= \frac{1}{(2B)^q} \sum_{n=0}^{q} \sum_{f \in \mathcal{F}_{n,q-n}} \sum_{\sigma \in S_{n,q-n}[f]} \rho_{\frac{1}{2}}(\sigma) \\ &= \frac{1}{(2B)^q} \sum_{n=0}^{q} \sum_{f \in \mathcal{F}_{n,q-n}} \sum_{\tau \in S_{n,q-n}} \rho_{\frac{1}{2}}(f \tau f^{-1}) \\ &= \frac{1}{(2B)^q} \sum_{n=0}^{q} \binom{q}{n} \sum_{\tau \in S_{n,q-n}} \rho_{\frac{1}{2}}(\tau) \end{aligned} \tag{4.74}$$

In the last equality, the binomial factor corresponds to the cardinality of $\mathcal{F}_{n,q-n}$. Finally, Lemma C.6 yields

$$M_\beta^{\mathrm{MF}}(q) = \frac{q!}{(2B)^q}, \tag{4.75}$$

which holds for both the unitary and orthogonal symmetry classes.

Autocorrelation Function of Degree 2

Let us consider the mean field autocorrelation functions

$$\left\langle |a_\alpha|^2 |a_{\alpha'}|^2 \right\rangle^{\mathrm{MF}} \equiv C^{\mathrm{MF}}_{[\alpha,\alpha']}, \tag{4.76}$$

where $\alpha \neq \alpha'$.

If $\alpha \neq \hat{\alpha}'$, it is easy to see from the general formula (4.65) that the unitary and orthogonal symmetry cases are given by the same expression, which reads

$$C^{\mathrm{MF}}_{[\alpha,\alpha']} = \frac{1}{(2B)^2} \sum_{t \in \{\uparrow,\downarrow\}^2} \rho_{\frac{1}{2}}(\mathrm{id}) = \frac{1}{(2B)^2}. \tag{4.77}$$

Suppose now that $\alpha = \hat{\alpha}'$. In the formula (4.65) for the unitary case, the identity permutation is still the only one to contribute. The result is therefore also given by (4.77). The situation is however changed in the orthogonal symmetry class. The identity permutation also brings the contribution (4.77),

but the transposition (0 1) brings an additional term

$$\frac{1}{(2B)^2}\sum_{t_0\in\{\uparrow,\downarrow\}}\rho_{\frac{1}{2}}(0\ 1)=\frac{1}{(2B)^2}. \tag{4.78}$$

Notice that in this last formula, only one of the two components t_0, t_1 is free and summed over since $\sigma = (0\ 1)$ forces t_0 and t_1 to differ from each other. In fact, it is not by chance that $\sigma = \mathrm{id}$ and $\sigma = (0\ 1)$ yield the same contribution. Indeed, the number of free components t_j to be summed over $\{\uparrow,\downarrow\}$ corresponds to the number of cycles in σ, and hence, the sum over these free components, providing a factor $2^{\text{number of cycles}}$ exactly compensates the factor $\rho_{\frac{1}{2}}(\sigma)$. Finally, we found that, for $\alpha \neq \alpha'$,

$$C^{\mathrm{MF}}_{[\alpha,\alpha']}=\frac{1}{(2B)^2}\begin{cases}1 & \text{(U)}\\ 1+\delta_{\alpha,\hat{\alpha}'} & \text{(O)}\end{cases} \tag{4.79}$$

Other Autocorrelation Functions

Let us finally compute the most general mean field autocorrelation function. It is convenient to come back to the notation used in Section 2.2 and write

$$\left\langle |a_{\beta_1}|^{2q_1}\dots|a_{\beta_n}|^{2q_n}\right\rangle^{\mathrm{MF}}\equiv C^{\mathrm{MF}}_{\boldsymbol{\beta}}(\boldsymbol{q}),\quad \text{for}\ \begin{cases}\boldsymbol{\beta}=(\beta_1,\dots,\beta_n)\\ \boldsymbol{q}=(q_1,\dots,q_n)\end{cases} \tag{4.80}$$

where $\beta_1,\dots,\beta_n$ are n different directed bonds. One can proceed by analogy with the first autocorrelation functions (4.76).

If the graph is unitary, the contributing permutations σ in (4.65) are those commuting the first q_1 elements together, the next q_2 together, and so forth. These permutations can be factorized into a product of n permutations, and the factor $\rho_{\frac{1}{2}}(\sigma)$ factorizes accordingly. This yields

$$C^{\mathrm{MF}}_{\boldsymbol{\beta}}(\boldsymbol{q})=\prod_{j=1}^{n}\frac{q_j!}{(2B)^{q_j}}. \tag{4.81}$$

This result also holds in the orthogonal case if there is no couple of reversed directed bonds in $\boldsymbol{\beta}$. However, this formula is modified if such couples exist in

β. Without loss of generality, one can suppose that

$$\begin{cases} \beta = (\beta_1, \ldots, \beta_m, \beta_{m+1}, \ldots, \beta_n) \\ q = (q_1, \ldots, q_m, q_{m+1}, \ldots, q_n) \end{cases}, \quad \begin{cases} \beta_j \equiv (\beta_j, \hat{\beta}_j) \\ q_j \equiv (q_j, \hat{q}_j) \end{cases}, \; j \in \mathbb{N}_m \tag{4.82}$$

for $0 \leq m \leq n$, and that the directed bonds in β are not supported by the same bond unless it is clearly indicated. Now, the contributing permutations are those which can be factorized

$$\sigma = (\sigma_1, \ldots, \sigma_m, \sigma_{m+1}, \ldots \sigma_n) \quad \text{with} \quad \sigma_j \in \begin{cases} S_{q_j+\hat{q}_j} & \text{if } 1 \leq j \leq m \\ S_{q_j} & \text{if } m < j \leq n \end{cases} \tag{4.83}$$

Here, the notation (σ, σ'), where $\sigma \in S_n$ and $\sigma' \in S_{n'}$, stands for the permutation in $S_{n+n'}$ where the first n elements are permuted with σ and the last n' elements are permuted with σ'. The generalization of this notation to a longer list of permutations, as in (4.83), is then obvious. The factorization (4.83) induces a corresponding factorization of the ρ factor in (4.65), and thus

$$C_\beta(q) = \prod_{j=1}^{m} C_{\beta_j}(q_j) \prod_{j=m+1}^{n} C_{\beta_j}(q_j). \tag{4.84}$$

Then, the key point is to realize that all the possible choices for a permutation σ_j in (4.83) contribute exactly the same amount to $C_\beta(q)$. The argument is totally similar to the one explained after (4.78) in the case $q = 2$. Indeed, if one permutation σ_j in (4.83) involves exactly k cycles, then there are exactly k free components t_j that have to be summed over $\{\uparrow, \downarrow\}$ in the general formula (4.65). These two factors cancel each other by $2^{-k}2^k = 1$. Hence, for β and q given by (4.82), we have, in the orthogonal symmetry case,

$$C_\beta^{\mathrm{MF}}(q) = \prod_{j=1}^{m} \frac{(q_j + \hat{q}_j)!}{(2B)^{q_j+\hat{q}_j}} \prod_{j=m+1}^{n} \frac{q_j!}{(2B)^{q_j}}. \tag{4.85}$$

Integral Formulation

In this paragraph, an alternative way to compute the mean field autocorrelation functions is presented. The idea is to express $C_\beta^{\mathrm{MF}}(q)$ as the derivative of

some determinant. From Theorem C.1, a short calculation shows that

$$C_\beta^{\mathrm{MF}}(q) = \frac{1}{(2B)^q}\left(\prod_{k=1}^{n}\frac{\partial^{q_k}}{\partial j_k^{q_k}}\right)\det\left[1-\sum_{k=1}^{n} j_k N(\beta_k)\right]^{-\frac{1}{2}}\Bigg|_{j=0}, \tag{4.86}$$

where, for $\beta \in \mathbb{N}_{2B}$, $N(\beta)$ is the matrix acting on $\mathcal{A}\otimes TR$ defined by

$$N(\beta) \equiv \begin{cases} E^{\beta,\beta}\otimes \mathbb{1}_{TR} & \text{(U)} \\ \begin{pmatrix} E^{\beta,\beta} & E^{\beta,\beta} \\ E^{\hat{\beta},\beta} & E^{\hat{\beta},\hat{\beta}} \end{pmatrix}_{TR} & \text{(O)} \end{cases} \tag{4.87}$$

Hence, the determinant in (4.86) generates all the mean field autocorrelation functions.

In the unitary symmetry case, the matrix in the determinant of (4.86) is trivial in time-reversal space. One can therefore remove this space and suppress the power $\frac{1}{2}$ of the determinant at the same time. The resulting determinant is easily computed, and one finds

$$C_\beta^{\mathrm{MF}}(q) = \frac{1}{(2B)^q}\prod_{k=1}^{n}\frac{\partial^{q_k}}{\partial j_k^{q_k}}\frac{1}{1-j_k}\Bigg|_{j_k=0} = \prod_{j=1}^{n}\frac{q_j!}{(2B)^{q_j}}, \tag{4.88}$$

which indeed coincides with the result (4.81) found with the permutation method presented previously.

Let us now consider the orthogonal case, and let us suppose β and q as in (4.82). Since the matrices $N(\beta)$ in (4.87) do not couple directed bonds that are not supported on the same bond, $C_\beta^{\mathrm{MF}}(q)$ in (4.86) factorizes as in (4.84). In this factorization, two kinds of quantities have to be computed. The first, $C_\beta^{\mathrm{MF}}(q)$, which actually corresponds to the moment $M_\beta^{\mathrm{MF}}(q)$, involves a single directed bond. The corresponding determinant in (4.86) can then simply be written as a determinant of a matrix acting on $V\otimes TR$, where $V \equiv \mathrm{vect}(|e_\beta\rangle, |e_{\hat{\beta}}\rangle)$ denotes the subspace of $\mathcal{A}$ spanned by $|e_\beta\rangle$ and $|e_{\hat{\beta}}\rangle$. More precisely, this determinant can be written

$$\det\left[1-j\left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)_V\right] = (1-j)^2. \tag{4.89}$$

Here, the inner 2×2 matrices act on the TR space. Plugging this result into the integral form (4.86) then yields

$$C_\beta^{\mathrm{MF}}(q) = \frac{1}{(2B)^q} \frac{\partial^q}{\partial j^q} \frac{1}{1-j}\bigg|_{j=0} = \frac{q!}{(2B)^q} \tag{4.90}$$

as wanted. The second kind of quantities that has to be computed is $C_{\beta,\hat{\beta}}^{\mathrm{MF}}(q,\hat{q})$. The integral formula (4.86) requires the introduction of two sources j and $\hat{j}$, and the corresponding determinant reads

$$\det\left[1 - j\left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)_V - \hat{j}\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)_V\right] = (1-j-\hat{j})^2. \tag{4.91}$$

Therefore, (4.86) gives

$$C_{\beta,\hat{\beta}}^{\mathrm{MF}}(q,\hat{q}) = \frac{1}{(2B)^{q+\hat{q}}} \frac{\partial^q}{\partial j^q} \frac{\partial^{\hat{q}}}{\partial j^{\hat{q}}} \frac{1}{1-j-\hat{j}}\bigg|_{j=\hat{j}=0} = \frac{(q+\hat{q})!}{(2B)^{q+\hat{q}}} \tag{4.92}$$

From the factorization (4.84) of $C_\beta^{\mathrm{MF}}(q)$, and from the formulae (4.90) and (4.92), we recover the result (4.85) for the orthogonal symmetry class.

4.3 The Crossed Convention

In Section 3.1, the generating function $\xi_{[\alpha]}(j)$ has been defined. The crucial property of this function is that its derivatives yield the autocorrelation functions, as stated in Theorem 3.1. In particular, the first non-trivial autocorrelation functions read

$$C_{[\alpha,\alpha']} = \left\langle |a_\alpha|^2 |a_{\alpha'}|^2 \right\rangle = \lim_{\epsilon \to 0} \frac{\epsilon}{B} \delta\xi_{\alpha,\alpha'} \tag{4.93}$$

For these autocorrelation functions, the generating function $\xi_{[\alpha,\alpha']}(j_a, j_r)$ involves the source supermatrices

$$J_a(j_a) \equiv 1 + j_a E_B \otimes E^{\alpha,\alpha} \quad \text{and} \quad J_r(j_r) \equiv 1 + j_r E_B \otimes E^{\alpha',\alpha'}. \tag{4.94}$$

The purpose of this paragraph is to introduce another function $\xi^{\times}_{[\alpha,\alpha']}(j_a, j_r)$, whose derivatives also fulfills (4.93). Working with this new function instead of $\xi_{[\alpha,\alpha']}(j_a, j_r)$ will bring some additional insights in the next chapter. Let us first define

$$J^{\times}_a(j_a) \equiv 1 + j_a E_B \otimes E^{\alpha,\alpha'} \quad \text{and} \quad J^{\times}_r(j_r) \equiv 1 + j_r E_B \otimes E^{\alpha',\alpha}. \tag{4.95}$$

The difference with (4.94) is that, in (4.95), the directed bonds α and α' appear in both $J^{\times}_a$ and $J^{\times}_r$. The writing (4.95) will then be referred to as the *crossed convention*, and the usual definitions (4.94) as the *parallel convention*. If $J^{\times}_a$ and $J^{\times}_r$ are substituted for J_a and J_r in $\xi_{[\alpha,\alpha']}(j_a, j_r)$, one gets

$$\xi^{\times}_{[\alpha,\alpha']}(j_a, j_r) \equiv \left\langle \mathrm{sdet}^{-1}\left(1 - J^{\times}_r U_\epsilon\right)\left(1 - J^{\times}_a U^{\dagger}_\epsilon\right)\right\rangle_k. \tag{4.96}$$

The crossed generating function (4.96) satisfies the equation (4.93). In order to check this claim, the steps in the proof of Theorem 3.1 can be followed. First, $\xi^{\times}_{[\alpha,\alpha']}(j_a, j_r)$ is written

$$\xi^{\times}_{[\alpha,\alpha']}(j_a, j_r) = \left\langle \det\left[1 - j_a E^{\alpha,\alpha'}\left(G^{\dagger} - 1\right)\right]^{-1}\left[1 - j_r E^{\alpha',\alpha}\left(G - 1\right)\right]^{-1}\right\rangle_k, \tag{4.97}$$

and then, Theorem C.1 is applied and gives

$$\delta\xi^{\times}_{[\alpha,\alpha']} = \left\langle \left[G^{\dagger} - 1\right]_{\alpha',\alpha}\left[G - 1\right]_{\alpha,\alpha'}\right\rangle_k. \tag{4.98}$$

This expression easily leads to the formula (4.93) for $\delta\xi^{\times}_{[\alpha,\alpha']}$. Indeed, by Theorem 2.5, the spectral average of a product between components of G or $G^{\dagger}$ only is regular at $\epsilon = 0$. Besides, Theorem 2.3 states that the places of the right indices of G and $G^{\dagger}$ in (4.98) are irrelevant, and that this expression, once multiplied by ϵ/B yields the autocorrelation function (4.93). In other terms, one has

$$\delta\xi^{\times}_{[\alpha,\alpha']} = \left\langle G^{\dagger}_{\alpha',\alpha} G_{\alpha,\alpha'}\right\rangle_k \overset{\epsilon}{=} \left\langle G^{\dagger}_{\alpha,\alpha} G_{\alpha',\alpha'}\right\rangle_k = \delta\xi_{[\alpha,\alpha']} \tag{4.99}$$

where the second equality sign holds up to terms that are analytic in ϵ around the origin.

Let us emphasize that the two possibilities $\delta\xi_{[\alpha,\alpha']}$ and $\delta\xi^{\times}_{[\alpha,\alpha']}$ for the generating function correspond to the two possibilities (2.91) and (2.92) to express $C_{[\alpha,\alpha']}$ in terms of classical paths. For more general autocorrelation functions, these various representations originate from the various choices of permutation σ in Theorem 2.3.

The argument presented above shows that the crossed generating function $\xi^{\times}_{[\alpha,\alpha']}$ provides the same result as the original generating function $\xi_{[\alpha,\alpha']}$. It is however not obvious that, when the integration domain in Theorem 3.4 are restricted onto the zero mode, the resulting mean field generating functions $\xi^{\times\mathrm{MF}}_{[\alpha,\alpha']}$ and $\xi^{\mathrm{MF}}_{[\alpha,\alpha']}$ lead to the same mean field autocorrelation functions $C^{\times\mathrm{MF}}_{[\alpha,\alpha']}$ and $C^{\mathrm{MF}}_{[\alpha,\alpha']}$. We will now check that the equality

$$C^{\times\mathrm{MF}}_{[\alpha,\alpha']} = C^{\mathrm{MF}}_{[\alpha,\alpha']} \tag{4.100}$$

indeed holds, in both the unitary and orthogonal symmetry classes. In fact, in the expression (4.32) for $C^{\mathrm{MF}}_{[\alpha,\alpha']}$, the only quantity that is sensitive to the parallel or crossed conventions (4.94) or (4.95) is the factor involving α and α', that is the factor $F_{[\alpha,\alpha']}(t,t',\sigma)$. Within the parallel formalism, this factor is defined as in (4.28) by

$$F_{[\alpha,\alpha']}(t,t',\sigma) \equiv \sum_{\gamma_0,\gamma_1\in\mathrm{N}_{2B}} \left[\mathcal{E}^{\alpha,\alpha}\right]_{\substack{\gamma_0,\gamma_{\sigma(0)}\\ t,t}} \left[\mathcal{E}^{\alpha',\alpha'}\right]_{\substack{\gamma_1,\gamma_{\sigma(1)}\\ t',t'}}. \tag{4.101}$$

However, if the crossed convention is preferred, the projectors $E^{\alpha,\alpha}$ and $E^{\alpha',\alpha'}$ must be replaced with the matrices $E^{\alpha,\alpha'}$ and $E^{\alpha',\alpha}$ respectively. Hence, the corresponding factor becomes

$$F^{\times}_{[\alpha,\alpha']}(t,t',\sigma) \equiv \sum_{\gamma_0,\gamma_1\in\mathrm{N}_{2B}} \left[\mathcal{E}^{\alpha,\alpha'}\right]_{\substack{\gamma_0,\gamma_{\sigma(0)}\\ t,t}} \left[\mathcal{E}^{\alpha',\alpha}\right]_{\substack{\gamma_1,\gamma_{\sigma(1)}\\ t',t'}}. \tag{4.102}$$

The last two formulae can be made explicit, and one finds

$$F_{[\alpha,\alpha']}(t,t',\sigma) = \begin{cases} 1 & \text{if } \sigma = \mathrm{id} \\ \begin{cases} \delta_{\alpha,\alpha'} & \text{if } t = t' \\ \delta_{\alpha,\hat{\alpha}'} & \text{if } t \neq t' \end{cases} & \text{if } \sigma = (0\ 1) \end{cases} \tag{4.103}$$

in the parallel case, and

$$F^{\times}_{[\alpha,\alpha']}(t,t',\sigma) = \begin{cases} \delta_{\alpha,\alpha'} & \text{if } \sigma = \mathrm{id} \\ \begin{cases} 1 & \text{if } t = t' \\ \delta_{\alpha,\hat{\alpha}'} & \text{if } t \neq t' \end{cases} & \text{if } \sigma = (0\ 1) \end{cases} \tag{4.104}$$

in the crossed case. In order to proceed further in the proof of (4.100), the unitary and orthogonal symmetry classes have now to be separately treated.

In the unitary case, the factor $I_\pi(t,t',\sigma)$ in (4.49) forces $t = t'$ if $\sigma = (0\ 1)$, so that the Kronecker symbols $\delta_{\alpha,\hat{\alpha}}$ in (4.103) and (4.104) never occur. Therefore, the parallel convention leads to

$$C^{\mathrm{MF}}_{[\alpha,\alpha']} = \frac{1}{(2B)^2}\left[\rho_{\frac{1}{2}}(\mathrm{id}) \sum_{t,t'\in\{\uparrow,\downarrow\}} 1 + \rho_{\frac{1}{2}}(0\ 1) \sum_{t,t'\in\{\uparrow,\downarrow\}} \delta_{t,t'}\delta_{\alpha,\alpha'}\right], \tag{4.105}$$

whereas with the crossed convention

$$C^{\times\mathrm{MF}}_{[\alpha,\alpha']} = \frac{1}{(2B)^2}\left[\rho_{\frac{1}{2}}(\mathrm{id}) \sum_{t,t'\in\{\uparrow,\downarrow\}} \delta_{\alpha,\alpha'} + \rho_{\frac{1}{2}}(0\ 1) \sum_{t,t'\in\{\uparrow,\downarrow\}} \delta_{t,t'}\right]. \tag{4.106}$$

In each term of these two expressions, the ρ factor compensates the sum over the free components in time-reversal space. This is precisely the mechanism explained after (4.78). Finally, both conventions lead to the result

$$C^{\mathrm{MF}}_{[\alpha,\alpha']} = C^{\times\mathrm{MF}}_{[\alpha,\alpha']} = \frac{1+\delta_{\alpha,\alpha'}}{(2B)^2}, \tag{4.107}$$

which agrees with the results found in (4.75), (4.77) and (4.79).

In the orthogonal case, the contribution of $(t,t',(0\ 1))$ with $t \neq t'$ has just to be added to the unitary formula. Since from (4.105) and (4.106) this contribution is the same in the two conventions, the final results still coincide, and they read

$$C^{\mathrm{MF}}_{[\alpha,\alpha']} = C^{\times\mathrm{MF}}_{[\alpha,\alpha']} = \frac{1+\delta_{\alpha,\alpha'}+\delta_{\alpha,\hat{\alpha}'}}{(2B)^2} \tag{4.108}$$

which is also in agreement with the results found in (4.75), (4.77) and (4.79).

It seems that there is no particular reason for working with the crossed convention since it provides the same exact results and the same mean field

results as the parallel convention. In the following chapter, the exact formulae are approximated beyond the mean field calculations. It will be seen that, at this higher level, the parallel and crossed predictions will then differ from each other, and the crossed formulae will yield an easier interpretation.

4.4 Mean Field and Universality

The mean field autocorrelation functions (4.81) and (4.85) for the unitary and orthogonal symmetry classes only depend on the quantum graph through its volume $2B$, and are thus universal results. Moreover, it is important to realize that these formulae agree with the universal Gaussian models (2.61) and (2.70), which are built from heuristic considerations in Section 2.2.

In particular, the mean field theory predicts asymptotic quantum ergodicity for any increasing sequence of quantum graphs. The mean field fluctuations of an observable V with $\bar{V} = 0$ read

$$\mathcal{F}_V^{\mathrm{MF}} = \kappa \frac{\mathrm{tr}(VL)^2}{(\mathrm{tr}L)^2}, \tag{4.109}$$

which also coincides with the fluctuations (2.47) coming from the long diagonal orbits formula. They decay in an universal way like B^{-1} as $B \longrightarrow \infty$, and they are twice larger if time-reversal invariance is conserved.

In [14], it is shown that the increasing sequence of star graphs is not quantum ergodic. The star graph with B bonds is the simple graph with $V = B+1$ vertices whose connectivity matrix satisfies $C_{1,j} = 1$ and $C_{j,k} = 0$ for any $j, k \in \mathbb{N}_V \setminus \{1\}$. Besides, in [12], some increasing sequences are proved to be ergodic. These indications show that the quantum ergodicity issue is not completely captured by the mean field theory.

In Chapter 5, the autocorrelation functions are approximated beyond the mean field level. The non-universal corrections obtained in this way give criteria for asymptotic quantum ergodicity to be met, or for the Gaussian models to apply, in which case the increasing sequence of graphs is said fully universal.

Chapter 5

The Gaussian Correction

5.1 Beyond Mean Field Theory

In Theorem 3.4, or equivalently in Corollary 3.5, the exact generating function $\xi_{[\alpha]}$ is represented as a superintegral over supermatrix variables Z and $\tilde{Z}$. In Chapter 4, a subset of the matrices $(Z, \tilde{Z})$, namely the zero mode, is isolated, and the integral over this subset provides the mean field generating function $\xi^{\mathrm{MF}}_{[\alpha]}$ in (4.10). In the same chapter, the mean field autocorrelation functions $C^{\mathrm{MF}}_{[\alpha]}$ are then obtained from substituting $\xi^{\mathrm{MF}}_{[\alpha]}$ for $\xi_{[\alpha]}$ in Theorem 3.1.

In this chapter, the exact generating function $\xi_{[\alpha]}$ is replaced with a *truncated generating function*

$$\xi_{[\alpha]}(j) \to \tilde{\xi}_{[\alpha]}(j) \equiv \xi^{\mathrm{MF}}_{[\alpha]}(j)\xi^{G}_{[\alpha]}(j), \tag{5.1}$$

where the *Gaussian generating function* $\xi^{G}_{[\alpha]}(j)$ is obtained from the exact superintegral representation in Corollary 3.5 by a Gaussian expansion of the action around the zero mode.

In fact, the Gaussian generating function is more easily obtained by first performing the superintegral in Corollary 3.5 over the whole second order action and then removing the contribution of the zero mode. This calculation

strategy reads

$$\xi^{G}_{[\alpha]}(j) = \frac{\xi^{(2)}_{[\alpha]}(j)}{\xi^{MF(2)}_{[\alpha]}(j)}, \tag{5.2}$$

where the *second order generating function* is written

$$\xi^{(2)}_{[\alpha]}(j) = \int d^{(2)}(Z,\tilde{Z})\; e^{-S^{(2)}[Z,\tilde{Z}]}. \tag{5.3}$$

The action $S^{(2)}[Z,\tilde{Z}]$ is obtained by developing the exact action $S[Z,\tilde{Z}]$ in Corollary 3.5 to second order around the zero mode configuration $(Z_0,\tilde{Z}_0) = (0,0)$, which corresponds to $Q = \sigma_3^{RA}$ in terms of Q matrices. It reads

$$S^{(2)}[Z,\tilde{Z}] = \mathrm{str}\left(Z\tilde{Z} - \frac{1}{2}\mathcal{J}_r\tilde{Z}^T\mathcal{J}_a\tilde{Z} - \frac{1}{2}ZS_\epsilon^\dagger Z^T S_\epsilon\right). \tag{5.4}$$

Notice that a true second order expansion of the exact action $S(Z_0+\delta Z,\tilde{Z}_0+\delta\tilde{Z})$ in the fields $(\delta Z,\delta\tilde{Z})$ orthogonal to the saddle-point manifold parametrized by $(Z_0,\tilde{Z}_0)$ would yield a decomposition $S^{MF}(Z_0,\tilde{Z}_0)+S^{(2)}(\delta Z,\delta\tilde{Z};Z_0,\tilde{Z}_0)$. In such an expansion, the second order action $S^{(2)}$ still depends on the saddle-point $(Z_0,\tilde{Z}_0)$ around which the second order is calculated. The calculation scheme described in (5.1)-(5.4) then corresponds to shift $(Z_0,\tilde{Z}_0)\to(0,0)$ in $S^{(2)}$. This is justified if the complete $(Z,\tilde{Z})$ manifold looks the same around each mean field configuration $(Z_0,\tilde{Z}_0)$, which we will assume. The measure $d^{(2)}(Z,\tilde{Z})$ is taken so that the superintegral gives one if all the sources are set to zero. It is the product of the Lebesgue measure for each commuting component and the Berezin measure for each anticommuting component. The second order mean field generating function is obtained by restricting the integral in (5.3) onto the zero mode, that is

$$\xi^{MF(2)}_{[\alpha]}(j) = \int d^{MF(2)}(Y)\; e^{-S^{MF(2)}[Y]}, \tag{5.5}$$

with

$$S^{MF(2)}[Y] = \frac{1}{2}\mathrm{str}\left((2-e^{-2\epsilon})\mathbb{1}_A\otimes Y\tilde{Y} - \mathcal{J}_r Y\mathcal{J}_a\tilde{Y}\right). \tag{5.6}$$

Then, by Theorem 3.1, the derivatives $\delta\tilde{\xi}_{[\alpha]}$, defined as in (3.11), lead to the formulae $\tilde{C}_{[\alpha]}$ for the autocorrelation functions. Property 4.2 states that if

the only source j_a is set to zero, the whole function $\xi^{\mathrm{MF}}_{[\alpha]}$ becomes identically equal to one, and the same happens to be true for $\xi^{(2)}_{[\alpha]}$ and $\xi^{\mathrm{MF}(2)}_{[\alpha]}$. Hence, if the only advanced derivative is first performed

$$\frac{\partial}{\partial j_a}\tilde{\xi}_{[\alpha]}(0,j_r) = \frac{\partial}{\partial j_a}\xi^{\mathrm{MF}}_{[\alpha]}(0,j_r) + \frac{\partial}{\partial j_a}\xi^{(2)}_{[\alpha]}(0,j_r) - \frac{\partial}{\partial j_a}\xi^{\mathrm{MF}(2)}_{[\alpha]}(0,j_r). \tag{5.7}$$

When a first retarded derivative is performed on (5.7), all the terms become singular at $\epsilon \to 0$. If $s \in \mathbb{N}$ retarded derivatives are performed on the mean field part of (5.7), then (4.32) and (4.33) show that the Laurent series start with a ϵ^{-s} term. The same will be true for the second order and the mean field second order generating functions. Hence, in the decomposition

$$\delta\tilde{\xi}_{[\alpha]} = \delta\xi^{\mathrm{MF}}_{[\alpha]} + \delta\xi^{(2)}_{[\alpha]} - \delta\xi^{\mathrm{MF}(2)}_{[\alpha]}, \tag{5.8}$$

all the terms behave like $\epsilon^{-(q-1)}$ if $[\alpha]$ is a list of q directed bonds. Together with Theorem 3.1, this identity leads to the *truncated autocorrelation functions*

$$\tilde{C}_{[\alpha]} = C^{\mathrm{MF}}_{[\alpha]} + C^{G}_{[\alpha]}, \tag{5.9}$$

where the mean field autocorrelation functions $C^{\mathrm{MF}}_{[\alpha]}$ are those obtained in Chapter 4, and the Gaussian autocorrelation functions $C^{G}_{[\alpha]}$ are defined by

$$C^{G}_{[\alpha]} \equiv \lim_{\epsilon\to 0} \frac{(2\epsilon)^{q-1}}{2B(q-1)!}\left[\delta\xi^{(2)}_{[\alpha]} - \delta\xi^{\mathrm{MF}(2)}_{[\alpha]}\right]. \tag{5.10}$$

In fact, the second order generating function $\xi^{(2)}_{[\alpha]}$ will be decomposed even further. The supermatrices Z and $\tilde{Z}$ introduced in (3.60) must be diagonal in the bond space $\mathcal{A}_b$ but do not need to be diagonal in the 2-dimensional direction space $\mathcal{A}_d$. The second order action $S^{(2)}[Z,\tilde{Z}]$ splits as

$$\begin{aligned} S^{(2)}[Z,\tilde{Z}] \;=\; & S^{(2)}_{(dd)}[Z^{\mathrm{diag}},\tilde{Z}^{\mathrm{diag}}] + S^{(2)}_{(oo)}[Z^{\mathrm{off}},\tilde{Z}^{\mathrm{off}}] \\ & + S^{(2)}_{(do)}[Z^{\mathrm{diag}},\tilde{Z}^{\mathrm{off}}] + S^{(2)}_{(od)}[Z^{\mathrm{off}},\tilde{Z}^{\mathrm{diag}}], \end{aligned} \tag{5.11}$$

where Z^{diag} and $\tilde{Z}^{\mathrm{diag}}$ denote the modes in Z and $\tilde{Z}$ that are diagonal in $\mathcal{A}_d$, and Z^{off} and $\tilde{Z}^{\mathrm{off}}$ denote the modes in Z and $\tilde{Z}$ that are off-diagonal in $\mathcal{A}_d$. In Section 5.2, the off-diagonal modes, and hence the actions $S^{(2)}_{(oo)}$, $S^{(2)}_{(do)}$ and $S^{(2)}_{(od)}$,

are discarded, and the integration (5.3) yields a generating function $\xi^{(2)}_{[\alpha],(dd)}$. In Section 5.3, the full second order action (5.11) is taken into account, and the result of the superintegral reads

$$\xi^{(2)}_{[\alpha]}(j) = \xi^{(2)}_{[\alpha],(dd)}(j) \cdot \xi^{(2)}_{[\alpha],(oo)}(j) \cdot \xi^{(2)}_{[\alpha],(do)}(j), \tag{5.12}$$

where $\xi^{(2)}_{[\alpha],(oo)}$ comes from only retaining the off-diagonal modes Z^{off} and $\tilde{Z}^{\text{off}}$ in the superintegral. The generating function $\xi^{(2)}_{[\alpha],(dd)}$ contains a contribution of the zero mode that has to be removed. By contrast, the (oo) and (dd) functions have no mean field components since both Y and $\tilde{Y}$ in (5.6) are diagonal in direction space. The Gaussian generating function (5.2) can thus be written

$$\xi^{G}_{[\alpha]}(j) = \xi^{G}_{[\alpha],(dd)}(j) \cdot \xi^{G}_{[\alpha],(oo)}(j) \cdot \xi^{G}_{[\alpha],(do)}(j), \tag{5.13}$$

with

$$\xi^{G}_{[\alpha],(dd)}(j) = \frac{\xi^{(2)}_{[\alpha],(dd)}(j)}{\xi^{\mathrm{MF}(2)}_{[\alpha]}(j)}, \tag{5.14}$$

and

$$\xi^{G}_{[\alpha],(oo)}(j) = \xi^{(2)}_{[\alpha],(oo)}(j), \quad \xi^{G}_{[\alpha],(do)}(j) = \xi^{(2)}_{[\alpha],(do)}(j). \tag{5.15}$$

When the derivative of $\xi^{G}_{[\alpha]}(j)$ with respect to the advanced source j_a is performed and evaluated at $j_a = 0$, the contributions of the four second order generating functions involved in (5.13) are additive as in (5.7). Besides, if s retarded derivatives are taken on the (dd) or on the (do) Gaussian generating function, the result, evaluated at $j_r = 0$, has a Laurent series starting with ϵ^{-s}. By contrast, taking retarded derivatives on the (oo) generating function does not create any singularity at $\epsilon = 0$. Hence,

$$\epsilon^{q-1}\delta\xi^{G}_{[\alpha]} = \epsilon^{q-1}\left[\delta\xi^{G}_{[\alpha],(dd)} + \delta\xi^{G}_{[\alpha],(do)}\right] + \mathcal{O}(\epsilon), \tag{5.16}$$

and

$$\epsilon^{q-1}\delta\xi^{G}_{[\alpha],(dd)} = \epsilon^{q-1}\left[\delta\xi^{(2)}_{[\alpha],(dd)} - \delta\xi^{\mathrm{MF}(2)}_{[\alpha]}\right] + \mathcal{O}(\epsilon). \tag{5.17}$$

Together with the definition (5.10), the equation (5.16) shows that the Gaussian autocorrelation also splits

$$C^{G}_{[\alpha]} = C^{G}_{[\alpha],(dd)} + C^{G}_{[\alpha],(do)}, \tag{5.18}$$

where

$$C^G_{[\alpha],(dd)} \equiv \lim_{\epsilon\to 0} \frac{(2\epsilon)^{q-1}}{2B(q-1)!}\left[\delta\xi^{(2)}_{[\alpha],(dd)} - \delta\xi^{\mathrm{MF}(2)}_{[\alpha]}\right], \tag{5.19}$$

and

$$C^G_{[\alpha],(do)} \equiv \lim_{\epsilon\to 0} \frac{(2\epsilon)^{q-1}}{2B(q-1)!}\delta\xi^{(2)}_{[\alpha],(do)}. \tag{5.20}$$

5.2 Diagonal Modes in Direction Space

5.2.1 Second Order Generating Function

In this section, only the modes Z and $\tilde{Z}$ that are diagonal in the amplitude space $\mathcal{A}$ are considered. These modes are written Z^{diag} and $\tilde{Z}^{\mathrm{diag}}$ in (5.11) but, for notation convenience, they are merely denoted by Z and $\tilde{Z}$ throughout this section. The second order superintegral (5.3) restricted to the diagonal modes then yields the generating function $\xi^{(2)}_{[\alpha],(dd)}$. The diagonal modes Z and $\tilde{Z}$ are parametrized in time-reversal space as follows,

$$Z = \begin{pmatrix} Z_1 & Z_2^\dagger \\ Z_3^{\mathcal{T}}\sigma_3^{\mathrm{BF}} & Z_4^{\dagger\mathcal{T}} \end{pmatrix} \quad \text{and} \quad \tilde{Z} = \begin{pmatrix} \tilde{Z}_1 & \sigma_3^{\mathrm{BF}}\tilde{Z}_3^{\mathcal{T}} \\ \tilde{Z}_2^\dagger & \tilde{Z}_4^{\dagger\mathcal{T}} \end{pmatrix}, \tag{5.21}$$

and their generalized transposes read

$$Z^\tau = \begin{pmatrix} Z_4^\dagger & \sigma_3^{\mathrm{BF}}Z_2^{\dagger\mathcal{T}} \\ Z_3 & Z_1^{\mathcal{T}} \end{pmatrix} \quad \text{and} \quad \tilde{Z}^\tau = \begin{pmatrix} \tilde{Z}_4^\dagger & \tilde{Z}_3 \\ \tilde{Z}_2^{\dagger\mathcal{T}}\sigma_3^{\mathrm{BF}} & \tilde{Z}_1^{\mathcal{T}} \end{pmatrix}. \tag{5.22}$$

The modes $Z_2^\dagger$, $\tilde{Z}_2^\dagger$, Z_3 and $\tilde{Z}_3$ only exist if time-reversal invariance is conserved. Let κ be the parameter such that $\kappa = 1$ if this symmetry is broken, and $\kappa = 2$ if it is conserved.

It is straightforward to check that the color-flavor requirements $\tilde{Z}_{\mathrm{BB}} = Z^\dagger_{\mathrm{BB}}$ and $\tilde{Z}_{\mathrm{FF}} = -Z^\dagger_{\mathrm{FF}}$ imply that

$$Z_j = \begin{pmatrix} Z_{j\mathrm{BB}} & Z_{j\mathrm{BF}} \\ Z_{j\mathrm{FB}} & Z_{j\mathrm{FF}} \end{pmatrix} \quad \text{and} \quad \tilde{Z}_j = \begin{pmatrix} Z^*_{j\mathrm{BB}} & \tilde{Z}_{j\mathrm{BF}} \\ \tilde{Z}_{j\mathrm{FB}} & -Z^*_{j\mathrm{FF}} \end{pmatrix}, \tag{5.23}$$

for any $j \in \mathbb{N}_4$. Each block of these matrices is $2B \times 2B$ and diagonal.

When the parametrizations (5.21) and (5.22) are plugged into (5.4), the diagonal modes in time-reversal space, which are indexed by 1 and 4, are coupled together, and do not mix with the off-diagonal ones indexed by 2 and 3. As a consequence, the restriction of the second order action $S^{(2)}$ to the diagonal modes can be decomposed as

$$S^{(2)}_{(dd)} = S^{(2)D}_{(dd)} + (\kappa - 1) S^{(2)C}_{(dd)} \tag{5.24}$$

where the *diffusion action* $S^{(2)D}_{(dd)}$ contains the modes indexed by 1 and 4 only, whereas the *cooperon action* $S^{(2)C}_{(dd)}$ only involves the modes indexed by 2 and 3. Hence, the generating function $\xi^{(2)}_{[\alpha],(dd)}$ factorizes as

$$\xi^{(2)}_{[\alpha],(dd)} = \begin{cases} \xi^{(2)D}_{[\alpha],(dd)} & \text{if } \kappa = 1 \\ \xi^{(2)D}_{[\alpha],(dd)} \xi^{(2)C}_{[\alpha],(dd)} & \text{if } \kappa = 2 \end{cases} \tag{5.25}$$

where, for $\circ = D, C$,

$$\xi^{(2)\circ}_{[\alpha],(dd)} = \int d^{(2)\circ}_{(dd)}(Z, \tilde{Z})\, e^{-S^{(2)\circ}_{(dd)}[Z,\tilde{Z}]}. \tag{5.26}$$

In (5.26), the measures $d^{(2)\circ}_{(dd)}(Z, \tilde{Z})$ are

$$\begin{aligned} d^{(2)D}_{(dd)}(Z, \tilde{Z}) \;=\; & \prod_{\beta=1}^{2B} \left(\prod_{j=1,4} \frac{dZ^*_{j\mathrm{BB}\beta} dZ_{j\mathrm{BB}\beta}}{\pi} \frac{dZ^*_{j\mathrm{FF}\beta} dZ_{j\mathrm{FF}\beta}}{\pi} \right) \\ & \cdot dZ_{1\mathrm{BF}\beta} dZ_{1\mathrm{FB}\beta} d\tilde{Z}_{1\mathrm{BF}\beta} d\tilde{Z}_{1\mathrm{FB}\beta} \\ & \cdot dZ^*_{4\mathrm{BF}\beta} dZ^*_{4\mathrm{FB}\beta} d\tilde{Z}^*_{4\mathrm{BF}\beta} d\tilde{Z}^*_{4\mathrm{FB}\beta} \end{aligned} \tag{5.27}$$

and

$$\begin{aligned} d^{(2)C}_{(dd)}(Z, \tilde{Z}) \;=\; & \prod_{\beta=1}^{2B} \left(\prod_{j=2,3} \frac{dZ^*_{j\mathrm{BB}\beta} dZ_{j\mathrm{BB}\beta}}{\pi} \frac{dZ^*_{j\mathrm{FF}\beta} dZ_{j\mathrm{FF}\beta}}{\pi} \right) \\ & \cdot dZ^*_{2\mathrm{BF}\beta} dZ^*_{2\mathrm{FB}\beta} d\tilde{Z}^*_{2\mathrm{BF}\beta} d\tilde{Z}^*_{2\mathrm{FB}\beta} \\ & \cdot dZ_{3\mathrm{BF}\beta} dZ_{3\mathrm{FB}\beta} d\tilde{Z}_{3\mathrm{BF}\beta} d\tilde{Z}_{3\mathrm{FB}\beta}. \end{aligned} \tag{5.28}$$

After some algebra, the two actions $S^{(2)D}_{(dd)}$ and $S^{(2)C}_{(dd)}$ are found to be

$$S^{(2)D}_{(dd)} = \mathrm{str}\left(Z_1 \tilde{Z}_1 + Z_4^\dagger \tilde{Z}_4^\dagger - J_r \tilde{Z}_4^\dagger J_a \tilde{Z}_1 - Z_1 S_\epsilon^\dagger Z_4^\dagger S_\epsilon \right) \tag{5.29}$$

$$S^{(2)C}_{(dd)} = \mathrm{str}\left(Z_2^\dagger \tilde{Z}_2^\dagger + Z_3 \tilde{Z}_3 - J_r \tilde{Z}_3 J_a^T \tilde{Z}_2^\dagger - Z_2^\dagger S_\epsilon^{T\dagger} Z_3 S_\epsilon \right) \tag{5.30}$$

In order to perform the diffusion and cooperon integrals in (5.26), the supertraces in (5.29) and (5.30) must be explicitely expanded in Bose-Fermi space. Besides, from the expressions (5.27)-(5.30), it can be noticed that the cooperon second order generating function defined in (5.25), which only exists if $\kappa = 2$, that is if $S^{\mathcal{T}} = S$, can be obtained from the diffusion one by replacing J_a with $J_a^{\mathcal{T}}$. It is thus sufficient to calculate $\xi^{(2)D}_{[\alpha],(dd)}$, and the result for $\xi^{(2)C}_{[\alpha],(dd)}$ will then follow from this correspondence. Let us define for each $\beta \in \mathbb{N}_{2B}$

$$\begin{aligned} \tilde{z}_{1\bar{0}\beta} &= (Z^*_{1\mathrm{BB}}\,,\, Z^*_{1\mathrm{FF}})_\beta \quad , \quad z_{1\bar{0}\beta} = \begin{pmatrix} Z_{1\mathrm{BB}} \\ Z_{1\mathrm{FF}} \end{pmatrix}_\beta , \\ z^\dagger_{4\bar{0}\beta} &= (Z^*_{4\mathrm{BB}}\,,\, -Z^*_{4\mathrm{FF}})_\beta \quad , \quad \tilde{z}^\dagger_{4\bar{0}\beta} = \begin{pmatrix} Z_{4\mathrm{BB}} \\ -Z_{4\mathrm{FF}} \end{pmatrix}_\beta . \end{aligned} \tag{5.31}$$

The vectors $\tilde{z}_{1\bar{0}\beta}$ and $z_{1\bar{0}\beta}$ contain the commuting parameters of $\tilde{Z}_{1\beta}$ and $Z_{1\beta}$ respectively, and the vectors $z^\dagger_{4\bar{0}\beta}$ and $\tilde{z}^\dagger_{4\bar{0}\beta}$ contain those of $Z^\dagger_{4\beta}$ and $\tilde{Z}^\dagger_{4\beta}$. Similarly, the anticommuting variables of the diffusion action are arranged in the vectors

$$\begin{aligned} \tilde{z}_{1\bar{1}\beta} &= (\tilde{Z}_{1\mathrm{BF}}\,,\, \tilde{Z}_{1\mathrm{FB}})_\beta \quad , \quad z_{1\bar{1}\beta} = \begin{pmatrix} Z_{1\mathrm{BF}} \\ Z_{1\mathrm{FB}} \end{pmatrix}_\beta , \\ z^\dagger_{4\bar{1}\beta} &= (Z^*_{4\mathrm{BF}}\,,\, Z^*_{4\mathrm{FB}})_\beta \quad , \quad \tilde{z}^\dagger_{4\bar{1}\beta} = \begin{pmatrix} \tilde{Z}^*_{4\mathrm{BF}} \\ \tilde{Z}^*_{4\mathrm{FB}} \end{pmatrix}_\beta . \end{aligned} \tag{5.32}$$

Collecting the $2B$ row-vectors $\tilde{z}_{1\bar{0}\beta}$ (resp. $z^\dagger_{4\bar{0}\beta}$) together, one can write a larger row vector $\tilde{z}_{1\bar{0}}$ (resp. $z^\dagger_{4\bar{0}}$). The column-vectors $z_{1\bar{0}}$ and $\tilde{z}^\dagger_{4\bar{0}}$ are formed similarly from $z_{1\bar{0}\beta}$ and $\tilde{z}^\dagger_{4\bar{0}\beta}$, and one proceeds in the same way with the anticommuting variables in (5.32). Let us introduce the $2B \times 2B$ matrix s defined from the Bose-Bose blocks of the source supermatrices J_a and J_r by

$$s(j)_{\beta\beta'} \equiv J_a(j_a)_{\mathrm{BB},\beta'\beta} J_r(j_r)_{\mathrm{BB},\beta\beta'}. \tag{5.33}$$

This matrix has the properties

$$s(j_a, 0)^T = J_a(j_a), \quad s(0, j_r) = J_r(j_r), \tag{5.34}$$

and

$$s(j_a, j_r)_{\beta\beta'} = s(j_a, 0)_{\beta'\beta} s(0, j_r)_{\beta,\beta'}. \tag{5.35}$$

A direct expansion of (5.29) in Bose-Fermi space then leads to

$$S^{(2)D}_{(dd)\bar{0}} = (\tilde{z}_{1\bar{0}}, z^{\dagger}_{4\bar{0}}) \begin{pmatrix} \mathbf{1}_{\mathcal{A}} \otimes \mathbf{1}_{2\times 2} & -\begin{pmatrix} s(j_a j_r) & \\ & s(0,0) \end{pmatrix} \\ -M_\epsilon \otimes \mathbf{1}_{2\times 2} & \mathbf{1}_{\mathcal{A}} \otimes \mathbf{1}_{2\times 2} \end{pmatrix} \begin{pmatrix} z_{1\bar{0}} \\ \tilde{z}^{\dagger}_{4\bar{0}} \end{pmatrix} \tag{5.36}$$

for the part of $S^{(2)D}_{(dd)}$ involving the commuting variables, and

$$S^{(2)D}_{(dd)\bar{1}} = (\tilde{z}_{1\bar{1}}, z^{\dagger}_{4\bar{1}}) \begin{pmatrix} \mathbf{1}_{\mathcal{A}} \otimes \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) & \begin{pmatrix} s(j_a,0) & \\ & s(0 j_r) \end{pmatrix} \\ M_\epsilon \otimes \mathbf{1}_{2\times 2} & \mathbf{1}_{\mathcal{A}} \otimes \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \end{pmatrix} \begin{pmatrix} z_{1\bar{1}} \\ \tilde{z}^{\dagger}_{4\bar{1}} \end{pmatrix} \tag{5.37}$$

for the part involving the anticommuting variables. Notice that these formulae depend on the scattering matrix $S_\epsilon = e^{-\epsilon} S$ only through the classical map $M_\epsilon = e^{-2\epsilon} M$ it generates.

By (5.26), (5.27), (5.36) and (5.37), the superintegral defining $\xi^{(2)D}_{[\alpha],(dd)}$ is Gaussian. Therefore, the formula (B.30) for Gaussian superintegrals can be applied, and the quadratic forms (5.36) and (5.37) lead to

$$\xi^{(2)D}_{[\alpha],(dd)} = \frac{\det\left(\mathbf{1}_{\mathcal{A}} - s(j_a, 0) M_\epsilon\right) \det\left(\mathbf{1}_{\mathcal{A}} - s(0, j_r) M_\epsilon\right)}{\det\left(\mathbf{1}_{\mathcal{A}} - M_\epsilon\right) \det\left(\mathbf{1}_{\mathcal{A}} - s(j_a, j_r) M_\epsilon\right)}. \tag{5.38}$$

The cooperon generating function $\xi^{(2)C}_{[\alpha],(dd)}$ is obtained from (5.38) by replacing s with s^C defined by

$$s^C(j)_{\alpha\alpha'} \equiv J^T_a(j_a)_{BB,\alpha'\alpha} J_r(j_r)_{BB,\alpha\alpha'}. \tag{5.39}$$

It reads

$$\xi^{(2)C}_{[\alpha],(dd)} = \frac{\det\left(\mathbf{1}_{\mathcal{A}} - s^C(j_a, 0) M_\epsilon\right) \det\left(\mathbf{1}_{\mathcal{A}} - s^C(0, j_r) M_\epsilon\right)}{\det\left(\mathbf{1}_{\mathcal{A}} - M_\epsilon\right) \det\left(\mathbf{1}_{\mathcal{A}} - s^C(j_a, j_r) M_\epsilon\right)}. \tag{5.40}$$

From these explicit formulae, the following property is easily seen to hold.

Property 5.1 *For any j_a and j_r in a sufficiently small neighborhood of the origin, and for any $\circ \in \{D, C\}$,*

$$\xi^{(2)\circ}_{[\alpha],(dd)}(j_a, 0) = \xi^{(2)\circ}_{[\alpha],(dd)}(0, j_r) = 1. \tag{5.41}$$

In particular, the same is true for the product $\xi^{(2)D}_{[\alpha],(dd)} \xi^{(2)C}_{[\alpha],(dd)}$.

As a consequence, if the advanced derivative is taken on $\xi^{(2)}_{[\alpha],(dd)}$ and evaluated at $j_a = 0$, one gets

$$\begin{aligned}\frac{\partial}{\partial j_a}\xi^{(2)}_{[\alpha],(dd)}(0,j_r) &= \frac{\partial}{\partial j_a}\xi^{(2)D}_{[\alpha],(dd)}(0,j_r) + (\kappa-1)\frac{\partial}{\partial j_a}\xi^{(2)C}_{[\alpha],(dd)}(0,j_r) \\ &= \left[Q^D_{[\alpha],(dd)}(j_r) - Q^D_{[\alpha],(dd)}(0)\right] \\ &\quad +(\kappa-1)\left[Q^C_{[\alpha],(dd)}(j_r) - Q^C_{[\alpha],(dd)}(0)\right], \end{aligned} \tag{5.42}$$

where the function $Q^\circ_{[\alpha],(dd)}$ reads

$$\begin{aligned}Q^\circ_{[\alpha],(dd)}(j_r) &\equiv -\frac{\partial}{\partial j_a}\frac{\det\left(\mathbb{1}_{\mathcal{A}} - s^\circ(j_a,j_r)M_\epsilon\right)}{\det\left(\mathbb{1}_{\mathcal{A}} - s^\circ(0,j_r)M_\epsilon\right)}\Bigg|_{j_a=0} \\ &= \mathrm{tr}\left[\frac{\partial s^\circ}{\partial j_a}(0,j_r)M_\epsilon\frac{1}{1-s^\circ(0,j_r)M_\epsilon}\right]. \end{aligned} \tag{5.43}$$

The last equality is a consequence of Theorem C.4.

5.2.2 Autocorrelation Functions of Second Degree

The next step towards the calculation of the Gaussian autocorrelation functions $C^G_{[\alpha],(dd)}$ is to perform the retarded derivatives on (5.43). In this subsection, we only consider autocorrelation functions $C_{[\alpha,\alpha']}$ of degree two. The calculation scheme presented here is then carried further to functions of higher degrees in the next subsection. In Section 4.3, the formulae (4.94) and (4.95) introduce two different conventions for the source supermatrices $J_a(j_a)$ and $J_r(j_r)$, called parallel and crossed conventions. They are shown to provide the same exact and mean field autocorrelation functions. In this subsection, the Gaussian autocorrelation functions $C^G_{[\alpha,\alpha'],(dd)}$ generated by the diagonal modes are computed with both these conventions, and it is shown that the resulting expressions do not coincide.

Derivatives of the Second Order Generating Function

In the case of autocorrelation functions of degree 2, there is only one retarded source j_r. The derivative of (5.43) with respect to this source yields

$$\begin{aligned}\delta\xi^{(2)\circ}_{[\alpha,\alpha'],(dd)} &= \frac{\partial}{\partial j_r} Q^\circ_{[\alpha,\alpha'],(dd)}(0) \\ &= \mathrm{tr}\left[\frac{M_\epsilon}{1-M_\epsilon} s^\circ_r \frac{M_\epsilon}{1-M_\epsilon} s^\circ_a + \frac{M_\epsilon}{1-M_\epsilon} s^\circ_{ra}\right], \end{aligned} \tag{5.44}$$

where s°_r, s°_a and s°_{ra} respectively denote the derivatives of s° with respect to j_r, j_a, and j_r and j_a, all evaluated at $j_r = j_a = 0$.

The functions $s^\circ(j_r, j_a)$ depend on the convention chosen. For the parallel convention (4.94), the definitions (5.33) and (5.39) of $s^\circ(j_r, j_a)$ lead to the derivatives

$$\begin{aligned} s^D_a &= E^{\alpha,\alpha} & s^C_a &= E^{\hat\alpha,\hat\alpha} \\ s^D_r &= E^{\alpha',\alpha'} & s^C_r &= E^{\alpha',\alpha'} \\ s^D_{ra} &= \delta_{\alpha,\alpha'} E^{\alpha,\alpha} & s^C_{ra} &= \delta_{\hat\alpha,\alpha'} E^{\hat\alpha,\hat\alpha} \end{aligned} \tag{5.45}$$

If the crossed convention (4.95) is preferred, and if $\mathrm{diag}(X)$ denotes the matrix obtained from X by setting all its non-diagonal components to zero, one obtains

$$\begin{aligned} s^{\times D}_a &= \mathrm{diag}\left(E^{\alpha',\alpha}\right) & s^{\times C}_a &= \mathrm{diag}\left(E^{\hat\alpha',\hat\alpha}\right) \\ s^{\times D}_r &= \mathrm{diag}\left(E^{\alpha',\alpha}\right) & s^{\times C}_r &= \mathrm{diag}\left(E^{\alpha',\alpha}\right) \\ s^{\times D}_{ra} &= E^{\alpha',\alpha} & s^{\times C}_{ra} &= \delta_{\hat\alpha,\alpha'} E^{\alpha,\hat\alpha} \end{aligned} \tag{5.46}$$

Let us first consider the diffusion generating function $\delta\xi^{(2)D}_{[\alpha,\alpha'],(dd)}$. In the parallel convention, (5.45) plugged into (5.44) provides

$$\delta\xi^{(2)D}_{[\alpha,\alpha'],(dd)} = \delta_{\alpha,\alpha'} \left(\frac{M_\epsilon}{1-M_\epsilon}\right)_{\alpha,\alpha} + \left(\frac{M_\epsilon}{1-M_\epsilon}\right)_{\alpha,\alpha'} \left(\frac{M_\epsilon}{1-M_\epsilon}\right)_{\alpha',\alpha}, \tag{5.47}$$

while in the crossed convention, (5.46) rather leads to

$$\delta\xi^{\times(2)D}_{[\alpha,\alpha'],(dd)} \equiv \left(\frac{M_\epsilon}{1-M_\epsilon}\right)_{\alpha,\alpha'} + \delta_{\alpha,\alpha'} \left(\frac{M_\epsilon}{1-M_\epsilon}\right)_{\alpha,\alpha} \left(\frac{M_\epsilon}{1-M_\epsilon}\right)_{\alpha,\alpha}. \tag{5.48}$$

These formulae have a natural interpretation in terms of orbits on the graph. However, these orbits, which are followed with the classical map M, are different in the two formulae if $\alpha \neq \alpha'$. In the next subsection, the off-diagonal

orbits will not modify the parallel expression, but they will add a contribution to the crossed one.

The corresponding cooperon expressions are

$$\delta\xi^{(2)C}_{[\alpha,\alpha'],(dd)} = \delta_{\hat{\alpha},\alpha'}\left(\frac{M_\epsilon}{1-M_\epsilon}\right)_{\hat{\alpha},\hat{\alpha}} + \left(\frac{M_\epsilon}{1-M_\epsilon}\right)_{\hat{\alpha},\alpha'}\left(\frac{M_\epsilon}{1-M_\epsilon}\right)_{\alpha',\hat{\alpha}}, \tag{5.49}$$

with the parallel convention, and

$$\delta\xi^{\times(2)C}_{[\alpha,\alpha'],(dd)} \equiv \delta_{\alpha',\hat{\alpha}}\left(\frac{M_\epsilon}{1-M_\epsilon}\right)_{\hat{\alpha},\hat{\alpha}} + \delta_{\alpha,\alpha'}\left(\frac{M_\epsilon}{1-M_\epsilon}\right)_{\hat{\alpha},\alpha}\left(\frac{M_\epsilon}{1-M_\epsilon}\right)_{\alpha,\hat{\alpha}}. \tag{5.50}$$

with the crossed convention.

It is interesting to notice that these results coincide with those obtained from the diagonal approximation in Section 3.2. The calculation reported in Section 3.2 uses the parallel convention, which is why the formula (3.39) coincides with (5.47) and its cooperon counterpart coincides with (5.49). However, the same calculation with the crossed convention produces the results (5.48) and (5.50).

Mean Field Contribution

According to the calculation scheme presented in Section 5.1, the second order mean field generating function has to be calculated, and its derivatives have to be subtracted from the previous second order formulae. Let us temporarily allow for several retarded sources grouped in j_r. The mean field modes satisfy $\tilde{Y} = Y^\tau$, and can thus be parametrized

$$Y = \begin{pmatrix} Y_{\mathrm{D}} & Y_{\mathrm{C}} \\ \tilde{Y}_{\mathrm{C}}^T \sigma_3^{BF} & \tilde{Y}_{\mathrm{D}}^T \end{pmatrix} \quad \text{and} \quad \tilde{Y} = \begin{pmatrix} \tilde{Y}_{\mathrm{D}} & \sigma_3^{BF} Y_{\mathrm{C}}^T \\ \tilde{Y}_{\mathrm{C}} & Y_{\mathrm{D}}^T \end{pmatrix} \tag{5.51}$$

in time-reversal space. If the graph is in the unitary class, the cooperon modes Y_C and $\tilde{Y}_C$ in (5.51) vanish. The second order mean field action (5.6) splits into a diffusion and a cooperon parts, $S^{\mathrm{MF}(2)} = S^{\mathrm{MF}(2)D} + S^{\mathrm{MF}(2)C}$, where

$$S^{\mathrm{MF}(2)D} = \mathrm{str}\left(\left(2-e^{-2\epsilon}\right) Y_{\mathrm{D}}\tilde{Y}_{\mathrm{D}} \otimes \mathbb{1}_{\mathcal{A}} - J_r Y_{\mathrm{D}} J_a \tilde{Y}_{\mathrm{D}}\right) \tag{5.52}$$

$$S^{\mathrm{MF}(2)C} = \mathrm{str}\left(\left(2-e^{-2\epsilon}\right) Y_{\mathrm{C}}\tilde{Y}_{\mathrm{C}} \otimes \mathbb{1}_{\mathcal{A}} - J_r Y_{\mathrm{C}} J_a^T \tilde{Y}_{\mathrm{C}}\right). \tag{5.53}$$

These expressions can be developed in Bose-Fermi space, and the supermatrix in the resulting quadratic form has inverse superdeterminant

$$\xi^{\mathrm{MF}(2)\circ}_{[\alpha]} = \frac{\left(1 - e^{-2\epsilon} - \sigma^{\circ}(j_a, 0)\right)\left(1 - e^{-2\epsilon} - \sigma^{\circ}(0, j_r)\right)}{\left(1 - e^{-2\epsilon}\right)\left(1 - e^{-2\epsilon} - \sigma^{\circ}(j_a, j_r)\right)}, \tag{5.54}$$

where

$$\sigma^{\circ}(j_a, j_r) \equiv \frac{1}{2B} \sum_{\beta,\beta'} s^{\circ}(j_a, j_r)_{\beta,\beta'} - 1, \tag{5.55}$$

and s^D and s^C are the functions defined in (5.33) and (5.39). The second order mean field generating function is thus

$$\xi^{\mathrm{MF}(2)}_{[\alpha]} = \begin{cases} \xi^{\mathrm{MF}(2)D}_{[\alpha]} & \text{if } \kappa = 1 \\ \xi^{\mathrm{MF}(2)D}_{[\alpha]} \xi^{\mathrm{MF}(2)C}_{[\alpha]} & \text{if } \kappa = 2 \end{cases} \tag{5.56}$$

Notice that the expression (5.54) has the following property.

Property 5.2 *For any j_a and j_r in a sufficiently small neighborhood of the origin, and for any $\circ \in \{D, C\}$,*

$$\xi^{MF(2)\circ}_{[\alpha]}(j_a, 0) = \xi^{MF(2)\circ}_{[\alpha]}(0, j_r) = 1. \tag{5.57}$$

In particular, the same is true for the product $\xi^{MF(2)D}_{[\alpha]} \xi^{MF(2)C}_{[\alpha]}$.

Let us come back to autocorrelation functions of degree 2, for which a single retarded source has to be considered. The advanced and the retarded derivatives can be taken on $\xi^{\mathrm{MF}(2)\circ}_{[\alpha,\alpha']}$, and one easily gets

$$\delta\xi^{\mathrm{MF}(2)\circ}_{[\alpha,\alpha']} = \left(\frac{1}{1 - e^{-2\epsilon}}\right)^2 \sigma_r \sigma_a + \left(\frac{1}{1 - e^{-2\epsilon}}\right) \sigma_{ra}, \tag{5.58}$$

where the indices r and a denote the derivatives taken on σ, which are all evaluated at the origin. These derivatives read

$$\begin{aligned} \sigma^D_a &= \tfrac{1}{2B} & \sigma^C_a &= \tfrac{1}{2B} \\ \sigma^D_r &= \tfrac{1}{2B} & \sigma^C_r &= \tfrac{1}{2B} \\ \sigma^D_{ra} &= \tfrac{\delta_{\alpha,\alpha'}}{2B} & \sigma^C_{ra} &= \tfrac{\delta_{\alpha,\alpha'}}{2B} \end{aligned} \tag{5.59}$$

for the parallel convention, and

$$\begin{aligned} \sigma_a^{\times D} &= \tfrac{\delta_{\alpha,\alpha'}}{2B} \quad \sigma_a^{\times C} = \tfrac{\delta_{\alpha,\alpha'}}{2B} \\ \sigma_r^{\times D} &= \tfrac{\delta_{\alpha,\alpha'}}{2B} \quad \sigma_r^{\times C} = \tfrac{\delta_{\alpha,\alpha'}}{2B} \\ \sigma_{ra}^{\times D} &= \tfrac{1}{2B} \quad \sigma_{ra}^{\times C} = \tfrac{\delta_{\hat{\alpha},\alpha'}}{2B} \end{aligned} \tag{5.60}$$

for the crossed convention. Hence, for $\circ \in \{D, C\}$, (5.58) becomes

$$\delta\xi^{\mathrm{MF}(2)\circ}_{[\alpha,\alpha']} = \frac{\delta_{\alpha,\alpha'}}{2B}\left(\frac{1}{1-e^{-2\epsilon}}\right) + \frac{1}{(2B)^2}\left(\frac{1}{1-e^{-2\epsilon}}\right)^2 \tag{5.61}$$

$$= \frac{\delta_{\alpha,\alpha'}}{2B}\left(\frac{e^{-2\epsilon}}{1-e^{-2\epsilon}}\right) + \frac{1}{(2B)^2}\left(\frac{e^{-2\epsilon}}{1-e^{-2\epsilon}}\right)^2 + \frac{1}{(2B)^2\epsilon} + \mathcal{O}(1) \tag{5.62}$$

with the parallel convention, and

$$\delta\xi^{\times\mathrm{MF}(2)\circ}_{[\alpha,\alpha']} = \frac{\delta_{\circ,D}+\delta_{\circ,C}\delta_{\hat{\alpha},\alpha'}}{2B}\left(\frac{1}{1-e^{-2\epsilon}}\right) + \frac{\delta_{\alpha,\alpha'}}{(2B)^2}\left(\frac{1}{1-e^{-2\epsilon}}\right)^2 \tag{5.63}$$

$$= \frac{\delta_{\circ,D}+\delta_{\circ,C}\delta_{\hat{\alpha},\alpha'}}{2B}\left(\frac{e^{-2\epsilon}}{1-e^{-2\epsilon}}\right) + \frac{\delta_{\alpha,\alpha'}}{(2B)^2}\left(\frac{e^{-2\epsilon}}{1-e^{-2\epsilon}}\right)^2 + \frac{\delta_{\alpha,\alpha'}}{(2B)^2\epsilon} + \mathcal{O}(1) \tag{5.64}$$

with the crossed convention. The formulae (5.62) and (5.64), which are obtained from (5.61) and (5.63) by a Taylor expansion, will provide an interpretation of these zero mode results in the next paragraph.

Gaussian Autocorrelation Functions

The mean field contributions (5.61) and (5.63) have to be subtracted from the second order formulae (5.47)-(5.50). Let us first consider the parallel formulae. In Corollary 1.2 the matrix $M_\epsilon(1-M_\epsilon)^{-1}$ is decomposed as the sum of an uniform part $|1\rangle\langle 1|$, which is singular in ϵ, and a remaining regular matrix R. Making use of this corollary and of the formulae (5.47) and (5.61) , a direct calculation shows that the diffusion Gaussian autocorrelation functions $C^{G,D}_{[\alpha,\alpha'],(dd)}$ read

$$C^{G,D}_{[\alpha,\alpha'],(dd)} \equiv \lim_{\epsilon\to 0}\frac{\epsilon}{B}\left[\delta\xi^{(2)D}_{[\alpha,\alpha'],(dd)} - \delta\xi^{\mathrm{MF}(2)D}_{[\alpha,\alpha']}\right] \tag{5.65}$$

$$= \frac{R_{\alpha\alpha'}+R_{\alpha'\alpha}}{(2B)^2} - \frac{2}{(2B)^3}. \tag{5.66}$$

Let us now make a couple of important remarks which provide a method for computing the Gaussian quantities from the second order ones without having to consider the second order mean field integrals. An expression like (5.47) for $\delta\xi^{(2)D}_{[\alpha,\alpha'],(dd)}$ can be seen as the sum of several contributions. One of these contributions is obtained from choosing in one term of (5.47) the uniform or the massive components $|1\rangle\langle 1|$ or R of $M_\epsilon(1-M_\epsilon)^{-1}$ in each factor $M_\epsilon(1-M_\epsilon)^{-1}$. Notice that such a contribution behaves like $\left(\frac{e^{-2\epsilon}}{1-e^{-2\epsilon}}\right)^n$ if the uniform contribution is chosen n times. The crucial point is that the contributions where the uniform component is systematically chosen originate from the zero mode, as it can be seen in (5.47) and (5.62). Hence, removing the mean field contribution kills all such terms. Among the remaining contributions to $\delta\xi^{(2)D}_{[\alpha,\alpha'],(dd)}$, only the most singular ones can survive the limit $\epsilon \to 0$ providing the Gaussian autocorrelation functions (5.65). These contributions originate from the term of (5.47) of second degree in $M_\epsilon(1-M_\epsilon)^{-1}$, and they are found by choosing the massive component R exactly once. These observations set up a correspondence between zero mode and uniform component of $M_\epsilon(1-M_\epsilon)^{-1}$ and provide at the same time a method to remove the contributions of the zero mode without having to explicitly compute them.

There is however a small correction that has to be added to the previous construction, namely, the second ratio in (5.66). This additional contribution comes from the last term in (5.62), which in turns is due to the fact that the mean field second order integrals exhibit the uniform component of $(1-M_\epsilon)^{-1}$ rather than the uniform component of $M_\epsilon(1-M_\epsilon)^{-1}$. This apparently surprising conclusion originates from the relation $\tilde{Y} = Y^T$ fulfilled by the zero mode. Indeed, if this relation is relaxed, the corresponding second order mean field integrals exhibit the uniform component of $M_\epsilon(1-M_\epsilon)^{-1}$. Besides, if the general modes Z and $\tilde{Z}$ are constrained to satisfy $\tilde{Z} = Z^T$, the matrix occurring in the formulae for the derivatives of the generating functions is not $M_\epsilon(1-M_\epsilon)^{-1}$, as in (5.47)-(5.50), but rather $(1-M_\epsilon)^{-1}$. Notice however that the additional contribution in (5.66) is of higher order in B^{-1}, and, since one is eventually

interested in asymptotically large B, this term is of minor importance. The previous construction of the Gaussian autocorrelation functions can still be modified in order to take this lower order term into account. From an expression for the derivatives of a second order generating function, the contribution of the zero mode is obtained by systematically substituting $(1-e^{-2\epsilon})^{-1}|1\rangle\langle 1|$ for $M_\epsilon(1-M_\epsilon)^{-1}$.

Let us now turn to the cooperon Gaussian autocorrelation functions in the parallel formalism. If the graph is not time-reversal invariant, there are no cooperon modes, and hence $C^{G,C}_{[\alpha,\alpha'],(dd)} = 0$. If the graph is time-reversal invariant, (5.49) and (5.62) imply

$$C^{G,C}_{[\alpha,\alpha'],(dd)} \equiv \lim_{\epsilon\to 0}\frac{\epsilon}{B}\left[\delta\xi^{(2)C}_{[\alpha,\alpha'],(dd)} - \delta\xi^{\mathrm{MF}(2)C}_{[\alpha,\alpha']}\right] \tag{5.67}$$

$$= \frac{R_{\hat{\alpha}\alpha'} + R_{\alpha'\hat{\alpha}}}{(2B)^2} - \frac{2}{(2B)^3}. \tag{5.68}$$

It can be checked that this formulae also follow from (5.49) and the construction mentioned above. As in (5.66), the last term in (5.68) is of higher order in B^{-1} and is due to the fact that the second order mean field integral involves the uniform contribution of $(1-M_\epsilon)^{-1}$ instead of $M_\epsilon(1-M_\epsilon)^{-1}$.

Let us now investigate these autocorrelation functions within the crossed formalism. As before, the explicit formula (5.64) for the mean field derivatives $\delta\xi^{\times\mathrm{MF}(2)\circ}_{[\alpha,\alpha']}$ can be used to remove the mean field contribution from $\delta\xi^{\times(2)\circ}_{[\alpha,\alpha'],(dd)}$ in (5.48) and (5.50), or equivalently, this mean field contribution can directly be spotted in (5.48) and (5.50) using the association between mean field and uniform component of $(1-M_\epsilon)^{-1}$. With both methods, one easily gets

$$C^{\times G,D}_{[\alpha,\alpha'],(dd)} \equiv \lim_{\epsilon\to 0}\frac{\epsilon}{B}\left[\delta\xi^{\times(2)D}_{[\alpha,\alpha'],(dd)} - \delta\xi^{\times\mathrm{MF}(2)D}_{[\alpha,\alpha']}\right] \tag{5.69}$$

$$= \delta_{\alpha,\alpha'}\frac{2R_{\alpha\alpha}}{(2B)^2} - \delta_{\alpha,\alpha'}\frac{2}{(2B)^3}. \tag{5.70}$$

and, if the graph is orthogonal, the cooperon autocorrelations read

$$C^{\times G,C}_{[\alpha,\alpha'],(dd)} \equiv \lim_{\epsilon\to 0}\frac{\epsilon}{B}\left[\delta\xi^{\times(2)C}_{[\alpha,\alpha'],(dd)} - \delta\xi^{\times\mathrm{MF}(2)C}_{[\alpha,\alpha']}\right] \tag{5.71}$$

$$= \delta_{\alpha,\alpha'}\frac{R_{\hat{\alpha}\alpha} + R_{\alpha\hat{\alpha}}}{(2B)^2} - \delta_{\alpha,\alpha'}\frac{2}{(2B)^3}. \tag{5.72}$$

If the graph is unitary, $C^{\times G,C}_{[\alpha,\alpha'],(dd)} = 0$.

In summary, the expressions (5.66), (5.68), (5.70) and (5.72) imply that the Gaussian autocorrelation functions $C^{G}_{[\alpha,\alpha'],(dd)}$ read

$$C^{G}_{[\alpha,\alpha'],(dd)} = \frac{R_{\alpha\alpha'} + R_{\alpha'\alpha} + (\kappa - 1)\big(R_{\hat{\alpha}\alpha'} + R_{\alpha'\hat{\alpha}}\big)}{(2B)^2} - \frac{2\kappa}{(2B)^3} \tag{5.73}$$

in the parallel convention, and

$$C^{\times G}_{[\alpha,\alpha'],(dd)} = \delta_{\alpha,\alpha'} C^{G}_{[\alpha,\alpha'],(dd)} \tag{5.74}$$

in the crossed convention.

The formulae (5.73) and (5.74) show that the predictions of the parallel and crossed conventions do not agree at the Gaussian level, at least if only the diagonal modes are considered.

5.2.3 Autocorrelation Functions of Higher Degrees

In this subsection, the result (5.73) of the parallel convention is generalized to autocorrelation of arbitrarily high degree. Let $[\alpha] \equiv [\alpha_0, \ldots, \alpha_{q-1}]$ be a list of q non-necessarily different directed bonds, and let $j = (j_a, j_1, \ldots j_{q-1})^T$ be the vector containing the sources. In (5.42), the derivative with respect to j_a has already been taken on $\xi^{(2)\circ}_{[\alpha],(dd)}$ for $\circ \in \{D, C\}$. One has now to compute

$$\delta\xi^{(2)\circ}_{[\alpha],(dd)} = \prod_{k=1}^{q-1} \frac{\partial}{\partial j_k} Q^{\circ}_{[\alpha]}(0), \tag{5.75}$$

where $Q^{\circ}_{[\alpha]}$ is defined in (5.43). Then, the corresponding autocorrelation function are given by the formula

$$C^{G\circ}_{[\alpha],(dd)} = \lim_{\epsilon \to 0} \frac{(2\epsilon)^{q-1}}{2B(q-1)!} \delta\xi^{G\circ}_{[\alpha],(dd)} \tag{5.76}$$

where the Gaussian derivatives $\delta\xi^{G\circ}_{[\alpha],(dd)}$ are obtained by removing the zero mode contribution to $\delta\xi^{(2)\circ}_{[\alpha],(dd)}$.

Let us now take the $q-1$ retarded derivatives in (5.75). From (5.43), and if s_k stands for the derivative of $s(j)$ with respect to j_k evaluated at the origin,

an inductive argument on q shows that these derivatives satisfy

$$\epsilon^{q-1}\delta\xi^{(2)\circ}_{[\alpha],(dd)} = \epsilon^{q-1}\sum_{\sigma\in S_{q-1}} \mathrm{tr}\left[s_a \frac{M_\epsilon}{1-M_\epsilon}\prod_{k=1}^{q-1} s_{\sigma(k)}\frac{M_\epsilon}{1-M_\epsilon}\right] + r(\epsilon, M), \quad (5.77)$$

where r contains either $\mathcal{O}(\epsilon)$ terms or terms depending on M only through its uniform $|1\rangle\langle 1|$ component. Notice that the right-hand side of (5.77) contains a diverging term in ϵ^{-1}. However, this term will be seen to come from the zero mode, as in 5.2.2. Besides, the definitions (5.33) and (5.39) of the functions s and s^C lead to the derivatives

$$\begin{aligned} s_a &= E^{\alpha_0,\alpha_0} \quad s^C_a = E^{\hat{\alpha}_0,\hat{\alpha}_0} \\ s_k &= E^{\alpha_k,\alpha_k} \quad s^C_k = E^{\alpha_k,\alpha_k} \end{aligned} \quad (5.78)$$

for any $k \in \mathbb{N}_{q-1}$. Let us first consider the diffusion quantities. Plugging the diffusion derivatives in (5.78) into (5.77) then yields

$$\epsilon^{q-1}\delta\xi^{(2),D}_{[\alpha],(dd)} = \epsilon^{q-1}\sum_{\substack{\sigma\in S_q \\ \text{transitive}}}\prod_{k=0}^{q-1}\left(\frac{M_\epsilon}{1-M_\epsilon}\right)_{\beta_k,\beta_{\sigma(k)}} + r(\epsilon, M) \quad (5.79)$$

Here, the sum is over the transitive permutations of S_q, namely the permutations having one q-cycle.

The zero mode contribution to (5.79) has now to be extracted from this expression. From the observations made after (5.66), this contribution is found by replacing all the factors $M_\epsilon(1-M_\epsilon)^{-1}$ in (5.79) with $(1-e^{-2\epsilon})^{-1}|1\rangle\langle 1|$. Following this procedure, one gets

$$\begin{aligned} (2\epsilon)^{q-1}\delta\xi^{G,D}_{[\alpha],(dd)} &:= (2\epsilon)^{q-1}\left\{\frac{(q-1)!}{(2B)^q}\left(\frac{e^{-2\epsilon}}{1-e^{-2\epsilon}}\right)^q - \frac{(q-1)!}{(2B)^q}\left(\frac{1}{1-e^{-2\epsilon}}\right)^q\right. \\ &\left. + \frac{(q-2)!}{(2B)^{q-1}}\left(\frac{e^{-2\epsilon}}{1-e^{-2\epsilon}}\right)^{q-1}\sum_{\substack{k,l=0\\k\neq l}}^{q-1} R_{\alpha_k\alpha_l}\right\} + \mathcal{O}(\epsilon) \quad (5.80) \\ &= \frac{(q-2)!}{(2B)^{q-1}}\sum_{\substack{k,l=0\\k\neq l}}^{q-1} R_{\alpha_k\alpha_l} - \frac{q!}{(2B)^q} + \mathcal{O}(\epsilon) \quad (5.81) \end{aligned}$$

Let us briefly comment (5.80). The first term is obtained from (5.79) by choosing the uniform component of $M_\epsilon(1-M_\epsilon)^{-1}$ in all the factors. It contains

the whole divergency hidden in (5.79). The factor $(q-1)!$ is the number of transitive permutations in S_q. The last term of (5.80) corresponds to taking the massive part R of $M_\epsilon(1-M_\epsilon)^{-1}$ in exactly one factor of (5.79) and the uniform component in the $q-1$ other factors. This term behaves like $\epsilon^{-(q-1)}$ and is thus the important one for our purposes. The factor $(q-2)!$ is the number of transitive permutations σ in S_q satisfying $\sigma(k)=l$, for some fixed elements $k,l \in \{0,\dots q-1\}$. Finally, the second term of (5.80) is the contribution of the zero mode to (5.79), which has to be removed.

The diffusion autocorrelation functions can be deduced from (5.76) and (5.81). They read

$$C^{G,D}_{[\alpha],(dd)} = \frac{1}{(q-1)(2B)^q} \sum_{\substack{k,l=0 \\ k\neq l}}^{q-1} R_{\alpha_k\alpha_l} - \frac{q}{(2B)^{q+1}} \tag{5.82}$$

The corresponding cooperon autocorrelation function is non-zero if and only if time-reversal symmetry is conserved, in which case the derivatives (5.78) indicate that it can be found by substituting $\hat{\alpha}_0$ for α_0 in the diffusion identity (5.82). Notice that these results indeed specialize to (5.73) as $q=2$.

5.3 Off-Diagonal Modes in Direction Space

5.3.1 The Full Second Order Generating Function

In Section 5.2, only the modes Z and $\tilde{Z}$ that are diagonal in the direction space, and hence in the whole amplitude space $\mathcal{A}$, have been taken into account. Here, this hypothesis is relaxed, and the possible modifications of the previous formulae are investigated. One can keep the parametrization (5.21) of Z and $\tilde{Z}$ in time-reversal space, and hence the formulae (5.29) and (5.30) also hold in the presence of off-diagonal modes. This implies in particular that all the cooperon formulae, which exist only if time-reversal invariance is conserved, can be found from their diffusion counterparts by replacing $J_a(j_a)$ with $J_a(j_a)^T$. One can thus temporarily concentrate on the diffusion modes only. However,

the pure blocks $Z_{jss'}$, $s, s' \in \{\mathrm{B}, \mathrm{F}\}$, are in general not diagonal in the direction space, so that the color-flavor requirement (5.23) has to be replaced with

$$Z_j = \begin{pmatrix} Z_{j\mathrm{BB}} & Z_{j\mathrm{BF}} \\ Z_{j\mathrm{FB}} & Z_{j\mathrm{FF}} \end{pmatrix} \quad \text{and} \quad \tilde{Z}_j = \begin{pmatrix} Z^\dagger_{j\mathrm{BB}} & \tilde{Z}_{j\mathrm{BF}} \\ \tilde{Z}_{j\mathrm{FB}} & -Z^\dagger_{j\mathrm{FF}} \end{pmatrix}. \tag{5.83}$$

The idea is to distinguish between modes that are diagonal in direction space and those that are off-diagonal. For this purpose, let us introduce the convenient notations

$$Z^{\mathrm{diag}}_\beta \equiv Z_{\beta\beta} \quad \text{and} \quad Z^{\mathrm{off}}_\beta \equiv Z_{\beta\hat{\beta}}, \tag{5.84}$$

and similarly for $\tilde{Z}$. These definitions produce twice more modes, which are now diagonal in $\mathcal{A}$. The quadratic action couples diagonal modes with themselves, which is precisely the part treated in Section 5.2, off-diagonal modes with themselves, and diagonal modes with off-diagonal modes.

The integration scheme used here is similar to the one that leads to the explicit formula (5.38) for $\xi^{(2)D}_{[\alpha],(dd)}$ in terms of four determinants. Let us first focus on the commuting components Z_{jss} of the fields. The row and column vectors defined in (5.31), whose purpose is to write the diagonal action $S^{(2)D}_{(dd)}$ as a quadratic form, are adapted to the situation where the fields Z and $\tilde{Z}$ are doubled according to (5.84). Let us define

$$\tilde{w}_\beta = \left(\tilde{z}^{\mathrm{diag}}_{1\bar{0}}, z^{\dagger\mathrm{diag}}_{4\bar{0}}, \tilde{z}^{\mathrm{off}}_{1\bar{0}}, z^{\dagger\mathrm{off}}_{4\bar{0}}\right)_\beta \tag{5.85}$$

where $\tilde{z}^{\mathrm{diag}}_{1\bar{0}}$ and $z^{\dagger\mathrm{diag}}_{4\bar{0}}$ are formed with the diagonal modes of $\tilde{z}_{1\bar{0}}$ and $z^\dagger_{4\bar{0}}$ defined in (5.31), and $\tilde{z}^{\mathrm{off}}_{1\bar{0}}$ and $z^{\dagger\mathrm{off}}_{4\bar{0}}$ are formed with the off-diagonal ones. We proceed in the same way with the column vectors and introduce

$$w_\beta = \left(z^{\mathrm{diag}\,T}_{1\bar{0}}, \tilde{z}^{\dagger\mathrm{diag}\,T}_{4\bar{0}}, z^{\mathrm{off}\,T}_{1\bar{0}}, \tilde{z}^{\dagger\mathrm{off}\,T}_{4\bar{0}}\right)^T_\beta. \tag{5.86}$$

The supertrace of the diffusion quadratic action $S^{(2)D}$, which is given by the right-hand side of (5.29), can be explicitely written in Bose-Fermi space. One

finds

$$
\begin{aligned}
S^{(2)D} \;=\; \mathrm{str}\Bigg\{ & \begin{pmatrix} Z^{\dagger}_{1\mathrm{BB}} & \tilde{Z}_{1\mathrm{BF}} \\ \tilde{Z}_{1\mathrm{FB}} & -Z^{\dagger}_{1\mathrm{FF}} \end{pmatrix}_{b,dd'} \begin{pmatrix} Z_{1\mathrm{BB}} & Z_{1\mathrm{BF}} \\ Z_{1\mathrm{FB}} & Z_{1\mathrm{FF}} \end{pmatrix}_{b,d'd} \\
& + \begin{pmatrix} Z^{\dagger}_{4\mathrm{BB}} & Z^{\dagger}_{4\mathrm{FB}} \\ -Z^{\dagger}_{4\mathrm{BF}} & Z^{\dagger}_{4\mathrm{FF}} \end{pmatrix}_{b,dd'} \begin{pmatrix} Z_{4\mathrm{BB}} & \tilde{Z}^{\dagger}_{4\mathrm{FB}} \\ -\tilde{Z}^{\dagger}_{4\mathrm{BF}} & -Z_{4\mathrm{FF}} \end{pmatrix}_{b,d'd} \\
& - \begin{pmatrix} Z_{4\mathrm{BB}} & \tilde{Z}^{\dagger}_{4\mathrm{FB}} \\ -\tilde{Z}^{\dagger}_{4\mathrm{BF}} & -Z_{4\mathrm{FF}} \end{pmatrix}_{b_1,d_1d_2} \begin{pmatrix} s(j_a,0)^T & 0 \\ 0 & s(0,0)^T \end{pmatrix}_{b_1d_2,b_2d_3} \\
& \cdot \begin{pmatrix} Z^{\dagger}_{1\mathrm{BB}} & \tilde{Z}_{1\mathrm{BF}} \\ \tilde{Z}_{1\mathrm{FB}} & -Z^{\dagger}_{1\mathrm{FF}} \end{pmatrix}_{b_2,d_3d_4} \begin{pmatrix} s(0,j_r) & 0 \\ 0 & s(0,0) \end{pmatrix}_{b_2d_4,b_1d_1} \\
& - \begin{pmatrix} Z^{\dagger}_{4\mathrm{BB}} & Z^{\dagger}_{4\mathrm{FB}} \\ -Z^{\dagger}_{4\mathrm{BF}} & Z^{\dagger}_{4\mathrm{FF}} \end{pmatrix}_{b_1,d_1d_2} S_{\epsilon b_1d_2,b_2d_3} \\
& \cdot \begin{pmatrix} Z_{1\mathrm{BB}} & Z_{1\mathrm{BF}} \\ Z_{1\mathrm{FB}} & Z_{1\mathrm{FF}} \end{pmatrix}_{b_2,d_3d_4} S^{\dagger}_{\epsilon b_2d_4,b_1d_1} \Bigg\}. \qquad (5.87)
\end{aligned}
$$

Here, the summation over repeated indices is implicit. The expression (5.87) is a complex quadratic form in the commuting components of the fields Z and $\tilde{Z}$. It generalizes the formula (5.36) for the diagonal quadratic action $S^{(2)D}_{(dJ)}$. Indeed, this diagonal action is recovered if only the modes diagonal in direction space are retained from the full expression (5.87).

A careful inspection of (5.87) shows that the part $S^{(2)D}_0$ containing the commuting variables can be written

$$
S^{(2)D}_0 = \sum_{\beta,\beta'=1}^{2B} \tilde{w}_\beta \mathcal{B}_{\beta\beta'} w_{\beta'} \qquad (5.88)
$$

where the $16B \times 16B$ matrix $\mathcal{B}$ is of the form

$$
\mathcal{B} = \left(\begin{array}{cc|cc}
\mathbf{1}_A \otimes \mathbf{1}_{2\times 2} & -\begin{pmatrix} s(j_a,j_r) & \\ & s(0,0) \end{pmatrix} & 0 & -A(j_a,j_r) \\
-M_\epsilon \otimes \mathbf{1}_{2\times 2} & \mathbf{1}_A \otimes \mathbf{1}_{2\times 2} & -\mathcal{G}_\epsilon & 0 \\
\hline
0 & -B(j_a,j_r) & X & -C(j_a,j_r) \\
-\mathcal{H}_\epsilon & 0 & -\mathcal{K}_\epsilon & Y
\end{array}\right) \qquad (5.89)
$$

where A, B, C, $\mathcal{G}_\epsilon$, $\mathcal{H}_\epsilon$, $\mathcal{K}_\epsilon$, X and Y remain to be discovered, the function s is defined in (5.33), and $M_\epsilon = e^{-2\epsilon}M$ is the classical map. In particular, the block $(1,1)$ of $\mathcal{B}$ couples the diagonal modes together, and it indeed coincides with (5.36). The other blocks involve the off-diagonal modes, and are still to be calculated. In (5.89), some zeroes have however been anticipated in these blocks. They partly come from the lack of coupling between $\tilde{z}^{\text{diag}}_{1\bar{0}}$ and $z^{\text{off}}_{1\bar{0}}$ on one hand, and between $\tilde{z}^{\text{off}}_{1\bar{0}}$ and $z^{\text{diag}}_{1\bar{0}}$ on the other, in the quadratic action (5.87). Similarly, the lack of coupling between $z^{\dagger\text{diag}}_{4\bar{0}}$ and $\tilde{z}^{\dagger\text{off}}_{4\bar{0}\beta}$, and between $z^{\dagger\text{off}}_{4\bar{0}}$ and $\tilde{z}^{\dagger\text{diag}}_{4\bar{0}\beta}$, justify the other two anticipated zeroes in (5.89).

The matrices A, B and C live in $\text{End}(\mathcal{A}\otimes\mathbb{C}^2)$ where $\mathbb{C}^2$ is the space whose first component contains the Bose-Bose modes and whose second component contains the Fermi-Fermi modes. A, B and C all come from the source part of $S^{(2)D}$, that is the third term of (5.87), which occupies the third and fourth lines of this formula. Notice that, since the source matrices on the third and fourth lines of (5.87) are diagonal in Bose-Fermi space, the Bose-Bose components of the fields are coupled together and the Fermi-Fermi components of the fields are coupled together. Besides, the Fermi-Fermi couplings can be retrieved from the Bose-Bose ones by sending the sources to zero. Indeed, the minus sign coming from the supertrace in the fermionic couplings is compensated by the minus sign introduced in the definition (5.31) of $\tilde{z}^\dagger_{4\bar{0}}$. In order to discover the matrices A, B and C, each block of $\mathcal{B}$ is separately investigated. The results are obtained for any number $q-1 \geq 1$ of retarded components, and in the case $q=2$, the crossed convention introduced in (4.95) is also considered.

- $A(j_a, j_r)$ couples $\tilde{z}^{\text{diag}}_{1\bar{0}}$ and $\tilde{z}^{\dagger\text{off}}_{4\bar{0}}$ and can thus be found considering the terms on the third and fourth lines of (5.87) with $d_3 = d_4$ and $d_1 = \hat{d}_2$. The corresponding coupling between Bose-Bose components reads

$$\begin{aligned} & -Z_{4\text{BB}b_1,d_1\hat{d}_1}\, s(j_a,0)^T_{b_1\hat{d}_1,b_2d_3}\, (Z^\dagger_{1\text{BB}})_{b_2,d_3d_3}\, s(0,j_r)_{b_2d_3,b_1d_1} \\ = \; & -Z^{\text{diag}*}_{1\text{BB}b_2d_3}\, s(j_a,0)^T_{b_1\hat{d}_1,b_2d_3}\, s(0,j_r)_{b_2d_3,b_1d_1}\, Z^{\text{off}}_{4\text{BB}b_1d_1}, \end{aligned} \tag{5.90}$$

where the sums over repeated indices are implicit. Therefore, the matrix

$A(j_a, j_r)$ reads

$$A(j_a, j_r) = \begin{pmatrix} a(j_a, j_r) & 0 \\ 0 & a(0,0) \end{pmatrix}, \tag{5.91}$$

with

$$\begin{aligned} a(j_a, j_r)_{\beta\beta'} &= s(j_a, 0)^T_{\hat{\beta}'\beta} s(0, j_r)_{\beta\beta'} \\ &= \delta_{\hat{\beta}'\beta}\delta_{\beta\beta'} + \delta_{\beta\beta'} j_a E^{(a)}_{\hat{\beta}'\beta} + \delta_{\hat{\beta}'\beta} j_r E^{(r)}_{\beta\beta'} + j_a E^{(a)}_{\hat{\beta}'\beta} j_r E^{(r)}_{\beta\beta'} \\ &= \delta_{\beta\beta'} j_a E^{(a)}_{\hat{\beta}\beta} + \delta_{\hat{\beta}'\beta} j_r E^{(r)}_{\beta\hat{\beta}}. \end{aligned} \tag{5.92}$$

In order to get the last equality, we use the fact that both the parallel and the crossed conventions for $E^{(a)}$ and $E^{(r)}$ predict that a term of the type $E^{(a)}_{\hat{\beta}'\beta} E^{(r)}_{\beta\beta'}$ and its cooperon counterpart $E^{(a)}_{\hat{\beta}\beta'} E^{(r)}_{\beta\hat{\beta}'}$ vanish.

- $B(j_a, j_r)$ couples $\tilde{z}^{\text{off}}_{10}$ with $\tilde{z}^{\text{diag}\dagger}_{40}$ and one has thus to consider $d_3 = \hat{d}_4$ and $d_1 = d_2$ on the third and fourth lines of (5.87). The corresponding Bose-Bose terms are

$$\begin{aligned} & -Z_{4\text{BB}b_1,d_1d_1} s(j_a, 0)^T_{b_1d_1,b_2d_3} (Z^{\dagger}_{1\text{BB}})_{b_2,d_3d_3} s(0, j_r)_{b_2d_3),b_1d_1} \\ = \; & -Z^{\text{off}*}_{1\text{BB}b_2d_3} s(j_a, 0)^T_{b_1d_1,b_2d_3} s(0, j_r)_{b_2d_3,b_1d_1} Z^{\text{diag}}_{4\text{BB}b_1d_1}. \end{aligned} \tag{5.93}$$

Therefore, the matrix $B(j_a, j_r)$ can be written

$$B(j_a, j_r) = \begin{pmatrix} b(j_a, j_r) & 0 \\ 0 & b(0,0) \end{pmatrix}, \tag{5.94}$$

with

$$\begin{aligned} b(j_a, j_r)_{\beta\beta'} &= s(j_a, 0)^T_{\beta'\hat{\beta}} s(0, j_r)_{\beta\beta'} \\ &= \delta_{\beta'\hat{\beta}}\delta_{\beta\beta'} + \delta_{\beta\beta'} j_a E^{(a)}_{\beta'\hat{\beta}} + \delta_{\beta'\hat{\beta}} j_r E^{(r)}_{\beta\beta'} + j_a E^{(a)}_{\beta'\hat{\beta}} j_r E^{(r)}_{\beta\beta'} \\ &= \delta_{\beta\beta'} j_a E^{(a)}_{\beta\hat{\beta}} + \delta_{\beta'\hat{\beta}} j_r E^{(r)}_{\beta\hat{\beta}}. \end{aligned} \tag{5.95}$$

The last equality is obtained by the same argument as the one presented for the matrix $A(j_a, j_r)$.

- $C(j_a, j_r)$ couples $z_{4\bar{0}}^{\dagger\text{off}}$ with $\tilde{z}_{4\bar{0}}^{\dagger\text{off}}$ so that one has to consider $d_3 = -d_4$ and $d_1 = -d_2$ on the third and fourth lines of (5.87). The same procedure as the one used for A and B shows that

$$C(j_a, j_r) = \begin{pmatrix} c(j_a, j_r) & 0 \\ 0 & c(0,0) \end{pmatrix}. \tag{5.96}$$

where

$$c(j_a, j_r)_{\beta\beta'} = s(j_a, 0)^T_{\hat{\beta}'\hat{\beta}} s(0, j_r)_{\beta\beta'}. \tag{5.97}$$

The matrices $\mathcal{G}_\epsilon$, $\mathcal{H}_\epsilon$ and $\mathcal{K}_\epsilon$ also live in $\mathrm{End}(\mathcal{A} \otimes \mathbb{C}^2)$, but they come from the scattering part of the action, that is from the fourth term in (5.87), which occupies the fifth ad sixth lines of this formula. Let us calculate these matrices. Notice first that, since the matrices S and $S^\dagger$ are trivial in Bose-Fermi space, the scattering part of $S_{\bar{0}}^{(2)D}$ couples Bose-Bose components with Bose-Bose components, and Fermi-Fermi components with Fermi-Fermi components, and moreover these two types of couplings are the same, since the sign coming from the supertrace in the fermionic couplings is compensated by the sign introduced in the definition (5.31) of $z_{4\bar{0}}^\dagger$. Therefore $\mathcal{G}_\epsilon$, $\mathcal{H}_\epsilon$ and $\mathcal{K}_\epsilon$ are trivial in $\mathbb{C}^2$ and it is sufficient to look at the couplings between Bose-Bose components.

- $\mathcal{G}_\epsilon$ couples $z_{4\bar{0}}^{\dagger\text{diag}}$ and $z_{1\bar{0}}^{\text{off}}$. Hence, it can be found by setting $d_1 = d_2$ and $d_3 = \hat{d}_4$ in the fourth term of (5.87). The corresponding Bose-Bose expression reads

$$\begin{aligned} & -(Z^\dagger_{4\text{BB}})_{b_1,d_1d_1} S_{\epsilon b_1 d_1, b_2 d_3} Z_{1\text{BB}b_2,d_3\hat{d}_3} S^\dagger_{\epsilon b_2 \hat{d}_3, b_1 d_1} \\ = \; & -Z^{\text{diag}*}_{4\text{BB}b_1 d_1} S_{\epsilon b_1 d_1, b_2 d_3} S^*_{\epsilon b_1 d_1, b_2 \hat{d}_3} Z^{\text{off}}_{1\text{BB}b_2 d_3}, \end{aligned} \tag{5.98}$$

and one readily gets

$$\mathcal{G}_{\epsilon\beta,\beta'} = S_{\epsilon\beta,\beta'} S^*_{\epsilon\beta,\hat{\beta}'} \cdot \mathbb{1}_{2\times 2}. \tag{5.99}$$

In order for such a matrix element to be non-zero, the graph should be such that $o\beta = t\beta'$ and $o\beta = t\hat{\beta}' = o\beta'$. These conditions can be

simultaneously fulfilled only if β' is a loop. Since we only consider simple graphs,

$$\mathcal{G}_\epsilon = 0. \tag{5.100}$$

- $\mathcal{H}_\epsilon$ couples $z_{4\bar{0}}^{\text{toff}}$ with $z_{1\bar{0}}^{\text{diag}}$ and can thus be found by setting $d_1 = \hat{d}_2$ and $d_3 = d_4$ in the fourth term of (5.87). The relevant Bose-Bose expression reads

$$\begin{aligned} &-(Z^{\dagger}_{4\text{BB}})_{b_1,d_1\hat{d}_1} S_{\epsilon b_1\hat{d}_1,b_2d_3} Z_{1\text{BB}b_2,d_3d_3} S^{\dagger}_{\epsilon b_2d_3,b_1d_1} \\ = \; &-Z^{\text{off}\bullet}_{4\text{BB}b_1\hat{d}_1} S_{\epsilon b_1\hat{d}_1,b_2d_3} S^{\bullet}_{\epsilon b_1d_1,b_2d_3} Z^{\text{diag}}_{1\text{BB}b_2d_3}, \end{aligned} \tag{5.101}$$

and the matrix $\mathcal{H}$ is thus

$$\mathcal{H}_{\beta,\beta'} = S_{\epsilon\beta,\beta'} S^{\bullet}_{\epsilon\hat{\beta},\beta'} \cdot \mathbf{1}_{2\times 2}. \tag{5.102}$$

As previously, for such a matrix element to be non-zero, β should be a loop, and hence,

$$\mathcal{H}_\epsilon = 0. \tag{5.103}$$

- $\mathcal{K}_\epsilon$ couples $z_{4\bar{0}}^{\text{toff}}$ with $z_{1\bar{0}}^{\text{off}}$ and can thus be found by setting $d_1 = \hat{d}_2$ and $d_3 = \hat{d}_4$ in the fourth term of (5.87). Proceeding as before, one finds

$$\mathcal{K}_{\epsilon\beta\beta'} = S_{\epsilon\beta\beta'} S^{\bullet}_{\epsilon\hat{\beta}\hat{\beta}'} \cdot \mathbf{1}_{2\times 2} = K_{\epsilon\beta\beta'} \cdot \mathbf{1}_{2\times 2}, \tag{5.104}$$

where the matrix K is implicitely defined by the last equality. This is non-zero whenever $t\beta' = o\beta$ and $t\hat{\beta}' = o\hat{\beta}$. Since the graphs we consider are simple, one must have $\hat{\beta} = \beta'$, so that

$$K_{\epsilon\beta,\beta'} = \delta_{\hat{\beta},\beta'} S_{\epsilon\beta\hat{\beta}} S^{\bullet}_{\epsilon\hat{\beta}\beta}. \tag{5.105}$$

The matrices X and Y in B remain to be analyzed. These two matrices come from the first and second terms of (5.87) respectively when $d = \hat{d}'$. In these couplings, the Bose-Bose components are mixed together and the Fermi-Fermi components are mixed together, and these two types of couplings are

the same. The corresponding Bose-Bose term is

$$\begin{aligned}&(Z^{\dagger}_{1\mathrm{BB}})_{b,d\hat{d}}Z_{1\mathrm{BB}b,\hat{d}d}+(Z^{\dagger}_{4\mathrm{BB}})_{b,d\hat{d}}Z_{4\mathrm{BB}b,\hat{d}d}\\ =&\sum_{b,d}Z^{\mathrm{off}*}_{1\mathrm{BB}b\hat{d}}Z^{\mathrm{off}}_{1\mathrm{BB}b\hat{d}}+Z^{\mathrm{off}*}_{4\mathrm{BB}b\hat{d}}Z^{\mathrm{off}}_{4\mathrm{BB}b\hat{d}}.\end{aligned}\tag{5.106}$$

Hence,

$$X_{\beta\beta'}=Y_{\beta\beta'}=\delta_{\beta\beta'}\cdot\mathbb{1}_{2\times 2}.\tag{5.107}$$

Finally, for any simple graph, the matrix $\mathcal{B}$ defining the complex quadratic form $S^{(2)D}_{\tilde{0}}$ reads

$$\mathcal{B}=\left(\begin{array}{cc|cc}\mathbb{1}_{\mathcal{A}}\otimes\mathbb{1}_{2\times 2} & -\binom{s(j)}{\mathbb{1}_{\mathcal{A}}} & 0 & -\binom{a(j)}{0}\\ -M_\epsilon\otimes\mathbb{1}_{2\times 2} & \mathbb{1}_{\mathcal{A}}\otimes\mathbb{1}_{2\times 2} & 0 & 0\\ \hline 0 & -\binom{b(j)}{0} & \mathbb{1}_{\mathcal{A}}\otimes\mathbb{1}_{2\times 2} & -\binom{c(j)}{\mathbb{1}_{\mathcal{A}}}\\ 0 & 0 & -K_\epsilon\otimes\mathbb{1}_{2\times 2} & \mathbb{1}_{\mathcal{A}}\otimes\mathbb{1}_{2\times 2}\end{array}\right)\tag{5.108}$$

The determinant of $\mathcal{B}$ has to be investigated. Its diagonal blocks, mixing diagonal components together and off-diagonal components together, are denoted by $\mathcal{B}_{dd}$ and $\mathcal{B}_{oo}$ respectively. The block mixing the diagonal components of $\tilde{w}$ in (5.88) with the off-diagonal components of w is written $\mathcal{B}_{do}$, and finally the block mixing the off-diagonal components of $\tilde{w}$ with the diagonal components of w is $\mathcal{B}_{od}$. If $\mathcal{B}_{do}$ or $\mathcal{B}_{od}$ were zero, what they almost are, the determinant of $\mathcal{B}$ would merely be the product of the determinants of the diagonal blocks, and there would be no interaction between diagonal modes and off-diagonal modes, that is $\det\mathcal{B}=\det\mathcal{B}_{dd}\det\mathcal{B}_{oo}$ with

$$\det\mathcal{B}_{dd}=\det\Big(\mathbb{1}_{\mathcal{A}}-M_\epsilon\Big)\det\Big(\mathbb{1}_{\mathcal{A}}-s(j)M_\epsilon\Big),\tag{5.109}$$

$$\det\mathcal{B}_{oo}=\det\Big(\mathbb{1}_{\mathcal{A}}-K_\epsilon\Big)\det\Big(\mathbb{1}_{\mathcal{A}}-c(j)K_\epsilon\Big).\tag{5.110}$$

The actual situation is given in Lemma 5.4. In order to prove this lemma , the next result is first needed.

Lemma 5.3 *The matrix $K=\lim_{\epsilon\to 0}K_\epsilon$ satisfies the following properties.*

1. *Its square reads $(K^2)_{\beta\beta'}=\delta_{\beta\beta'}M_{\beta\hat{\beta}}M_{\hat{\beta}\beta}$.*

2. *Its spectrum $\sigma(K)$ is real and contained in the closed interval $[-1,1]$.*

3. *If the quantum graph contains more than one bond, and if no bond is dynamically disconnected from the rest, then $\sigma(K) \subset (-1,1)$.*

Proof. The first point is straightforward to check from the definition (5.105) of K_ϵ. Since, from the first point of the lemma, K^2 is diagonal in the basis $\{|e_\beta)\}_{\beta\in\mathbb{N}_{2B}}$ of the amplitude space, its $2B$ eigenvalues can directly be read on this formula. They read $M_{\beta\hat{\beta}}M_{\hat{\beta}\beta}$ for $\beta \in \mathbb{N}_{2B}$. Hence, the spectrum of K can be written $\sigma(K) = \{r_\beta\}_{\beta\in\mathbb{N}_{2B}}$, so that

$$r_\beta^2 = M_{\beta\hat{\beta}}M_{\hat{\beta}\beta} \tag{5.111}$$

for any $\beta \in \mathbb{N}_{2B}$. Since M is bistochastic, all its components are real and contained in $[0,1]$. Therefore, (5.111) shows that the eigenvalues r_β are also real, and $|r_\beta| \leq 1$. This proves the second point of the lemma.

In order to prove the third point of the lemma, it suffices to show that, for a dynamically connected graph, the right-hand side of (5.111) is smaller than unity. Let β be a directed bond. If $M_{\hat{\beta}\beta} < 1$, then the right-hand side of (5.111) is smaller than one, which implies the wanted result. Suppose now that $M_{\hat{\beta}\beta} = 1$. Then, for the directed bond b supporting β not to be dynamically disconnected, one must have $M_{\beta\hat{\beta}} < 1$. Indeed, suppose on the contrary that $M_{\beta\hat{\beta}} = 1$ is also satisfied. By assumption, one can consider another bond $b' \neq b$ and a directed bond β' supported on b'. Then, it is easy to see that $(e_{\beta'}|M^n|e_\beta) = 0$ for any $n \in \mathbb{N}$. This property holds for any directed bond β' supported on a bond $b' \neq b$. This precisely implies that the graph is dynamically disconnected, which contradicts the hypothesis.

□

Lemma 5.4 *The determinant of the matrix B in (5.108) reads*

$$\det B = \det B_{dd} \det B_{\infty\infty} \det\left(\mathbf{1}_{\mathcal{A}} - M_\epsilon \frac{1}{1-s(j)M_\epsilon} a(j) K_\epsilon \frac{1}{1-c(j)K_\epsilon} b(j)\right),$$

where the matrix valued functions $s(j)$, $c(j)$, $a(j)$ and $b(j)$ are defined in (5.33), (5.97), (5.92) and (5.95) respectively, and the determinants of $\mathcal{B}_{dd}$ and $\mathcal{B}_{oo}$ are given by the formulae (5.109) and (5.110).

Proof. One has to compute the determinant of a matrix having a 2×2 block structure. Applying Theorem D.2 in Appendix D to $\mathcal{B}$ leads to

$$\det \mathcal{B} = \det \mathcal{B}_{dd} \det \mathcal{B}_{oo} \det \left(\mathbb{1} - \mathcal{B}_{dd}^{-1} \mathcal{B}_{do} \mathcal{B}_{oo}^{-1} \mathcal{B}_{od} \right) . \tag{5.112}$$

The first two determinants of the right-hand side are precisely the first two determinants in the lemma, so that it is sufficient to investigate the last factor. For notation convenience, the indices ϵ and the sources j are not explicitly written. From (5.108), the matrices $\mathcal{B}_{dd}$ and $\mathcal{B}_{oo}$ and their inverses read

$$\begin{aligned} \mathcal{B}_{dd} &= \begin{pmatrix} \mathbb{1} & -\Sigma \\ -M & \mathbb{1} \end{pmatrix} \quad , \quad \mathcal{B}_{dd}^{-1} = \begin{pmatrix} \frac{1}{1-\Sigma M} & \Sigma \frac{1}{1-M\Sigma} \\ M \frac{1}{1-\Sigma M} & \frac{1}{1-M\Sigma} \end{pmatrix} \\ \mathcal{B}_{oo} &= \begin{pmatrix} \mathbb{1} & -C \\ -K & \mathbb{1} \end{pmatrix} \quad , \quad \mathcal{B}_{oo}^{-1} = \begin{pmatrix} \frac{1}{1-CK} & C \frac{1}{1-KC} \\ K \frac{1}{1-CK} & \frac{1}{1-KC} \end{pmatrix} \end{aligned} \tag{5.113}$$

where the unit matrices on the left-hand sides are $\mathbb{1}_{\mathcal{A}} \otimes \mathbb{1}_{2\times 2}$, the matrix C is defined by (5.96), and the matrix Σ is implicitly defined by comparison with the block (1,1) of (5.108). Besides, the matrices $\mathcal{B}_{do}$ and $\mathcal{B}_{od}$ can be written

$$\mathcal{B}_{do} = \begin{pmatrix} 0 & -A \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{B}_{od} = \begin{pmatrix} 0 & -B \\ 0 & 0 \end{pmatrix} , \tag{5.114}$$

where A and B are the matrices given in (5.91) and (5.94). Then, a direct calculation leads to

$$\mathcal{B}_{dd}^{-1} \mathcal{B}_{do} \mathcal{B}_{oo}^{-1} \mathcal{B}_{od} = \begin{pmatrix} 0 & \frac{1}{1-\Sigma M} AK \frac{1}{1-CK} \\ 0 & M \frac{1}{1-\Sigma M} AK \frac{1}{1-CK} B \end{pmatrix} \tag{5.115}$$

Therefore, the determinant to be computed reads

$$\det(\mathbb{1} - \mathcal{B}_{dd}^{-1} \mathcal{B}_{do} \mathcal{B}_{oo}^{-1} \mathcal{B}_{od}) = \det \left(\mathbb{1} - M \frac{1}{1-\Sigma M} AK \frac{1}{1-CK} B \right) . \tag{5.116}$$

Besides, by definition of A, B, C and Σ, the matrix occurring in the determinant of the right-hand side can be written

$$M\frac{1}{1-\Sigma M}AK\frac{1}{1-CK}B=\begin{pmatrix} M\frac{1}{1-sM}aK\frac{1}{1-cK}b & 0 \\ 0 & 0 \end{pmatrix} \tag{5.117}$$

Hence,

$$\det(\mathbb{1}-\mathcal{B}_{dd}^{-1}\mathcal{B}_{do}\mathcal{B}_{oo}^{-1}\mathcal{B}_{od})=\det\left(\mathbf{1}_{\mathcal{A}}-M\frac{1}{1-sM}aK\frac{1}{1-K}b\right), \tag{5.118}$$

which concludes the proof of the lemma

□

The full second order generating function $\xi^{(2)}_{[\alpha]}$ can actually be inferred from the determinant of $\mathcal{B}$ given in Lemma 5.4, and the result is exposed in the next lemma. Let us first define, for $o = D, C$, the functions

$$\xi^{(2)o}_{[\alpha],(dd)}(j_a,j_r)\equiv\frac{\det\left(\mathbf{1}_{\mathcal{A}}-s^{o}(j_a,0)M_\epsilon\right)\det\left(\mathbf{1}_{\mathcal{A}}-s^{o}(0,j_r)M_\epsilon\right)}{\det\left(\mathbf{1}_{\mathcal{A}}-M_\epsilon\right)\det\left(\mathbf{1}_{\mathcal{A}}-s^{o}(j_a,j_r)M_\epsilon\right)}, \tag{5.119}$$

$$\xi^{(2)o}_{[\alpha],(oo)}(j_a,j_r)\equiv\frac{\det\left(\mathbf{1}_{\mathcal{A}}-c^{o}(j_a,0)K_\epsilon\right)\det\left(\mathbf{1}_{\mathcal{A}}-c^{o}(0,j_r)K_\epsilon\right)}{\det\left(\mathbf{1}_{\mathcal{A}}-K_\epsilon\right)\det\left(\mathbf{1}_{\mathcal{A}}-c^{o}(j_a,j_r)K_\epsilon\right)} \tag{5.120}$$

and

$$\xi^{(2)o}_{[\alpha],(do)}(j_a,j_r)\equiv\frac{\det\left(\mathbf{1}_{\mathcal{A}}-N^{o}_\epsilon(j_a,0)\right)\det\left(\mathbf{1}_{\mathcal{A}}-N^{o}_\epsilon(0,j_r)\right)}{\det\left(\mathbf{1}_{\mathcal{A}}-N^{o}_\epsilon(0,0)\right)\det\left(\mathbf{1}_{\mathcal{A}}-N^{o}_\epsilon(j_a,j_r)\right)}, \tag{5.121}$$

where

$$N^{o}_\epsilon(j)\equiv M_\epsilon\frac{1}{1-s^{o}(j)M_\epsilon}a^{o}(j)K^{o}_\epsilon\frac{1}{1-c^{o}(j)K^{o}_\epsilon}b^{o}(j) \tag{5.122}$$

In these formulae, the cooperon quantities, indexed by C, are defined by substituting $J_a(j_a)^{\mathcal{T}}$ for $J_a(j_a)$ in their diffusion counterparts. Notice that $\xi^{(2)o}_{[\alpha],(dd)}$ is nothing else but the second order generating function found in (5.38) and (5.40) by taking only the diagonal modes into account. Let κ be 1 if time-reversal invariance is broken and 2 if this symmetry is conserved.

Lemma 5.5 *The second order generating function defined in (5.3) of a simple graph factorizes as*

$$\xi^{(2)}_{[\alpha]}(j) = \begin{cases} \xi^{(2)D}_{[\alpha]}(j) & \text{if } \kappa = 1 \\ \xi^{(2)D}_{[\alpha]}(j)\xi^{(2)C}_{[\alpha]}(j) & \text{if } \kappa = 2 \end{cases}$$

Furthermore, the diffusion ($\circ = D$) and cooperon ($\circ = C$) generating functions can be factorized as

$$\xi^{(2)\circ}_{[\alpha]}(j) = \xi^{(2)\circ}_{[\alpha],(dd)}(j)\xi^{(2)\circ}_{[\alpha],(oo)}(j)\xi^{(2)\circ}_{[\alpha],(do)}(j)$$

where these three functions are given by (5.119), (5.120) and (5.121).

Proof. The factorization of $\xi^{(2)}_{[\alpha]}$ into a diffusion and a cooperon generating functions has already been discussed at the beginning of this subsection. Moreover, it is clear from the expressions (5.29) and (5.30) for $S^{(2)D}$ and $S^{(2)C}$ that $\xi^{(2)C}_{[\alpha]}$ can be obtained from $\xi^{(2)D}_{[\alpha]}$ by replacing $J_a(j_a) \longrightarrow J_a(j_a)^{\mathcal{T}}$, which is exactly how the cooperon formulae in (5.119), (5.120) and (5.121) are obtained from their diffusion counterparts. It is thus sufficient to prove the factorization of $\xi^{(2)D}_{[\alpha]}$ presented in the lemma.

The contribution of the commuting variables in $S^{(2)D}$ has been the main focus in this subsection. In (5.88), these variables compose a complex quadratic form $S^{(2)D}_{\bar{0}}$ defined by a matrix $\mathcal{B}$, whose final expression is given in (5.108), and whose determinant obeys Lemma 5.4. The contribution of the anticommuting variables in $S^{(2)D}$ remains to be unveiled. The matrix defining the quadratic form $\mathcal{B}_{\bar{1}}$ mixing these anticommuting variables together must be deduced from the formula (5.87) for the action $S^{(2)D}$, and its determinant has to be computed. Then, the general formula (B.30) for Gaussian superintegrals implies that the second order generating function reads $\xi^{(2)D}_{[\alpha]} = \det \mathcal{B}_{\bar{1}} / \det \mathcal{B}$.

There is however a more clever way to deduce $\xi^{(2)}_{[\alpha]}$ from $\det \mathcal{B}$, which does not require any calculation. The matrix $\mathcal{B}$ is the matrix mixing the Bose-Bose components of the fields together and the Fermi-Fermi components together.

The formula (5.87) for the action $S^{(2)D}$ shows that the coupling between Fermi-Bose and Fermi-Bose components is obtained from the coupling between Bose-Bose and Bose-Bose components by setting $j_a = 0$ in the latter. Similarly, the coupling between Bose-Fermi and Bose-Fermi components is obtained from the coupling between Bose-Bose and Bose-Bose by setting $j_r = 0$. In fact, the coupling between Fermi-Fermi and Fermi-Fermi components is also found from the coupling between Bose-Bose and Bose-Bose components if all the sources are set to zero. From this last remark, and from Lemma 5.4, the contribution of the coupling between Bose-Bose and Bose-Bose components to $\det \mathcal{B}$ can easily be recognized. It reads

$$\begin{aligned} S^{(2)D}_{BB-BB} &= \det\left(\mathbf{1}_{\mathcal{A}} - s(j)M_\epsilon\right) \det\left(\mathbf{1}_{\mathcal{A}} - c(j)K_\epsilon\right) \\ &\quad \det\left(\mathbf{1}_{\mathcal{A}} - M_\epsilon \frac{1}{1-s(j)M_\epsilon} a(j) K_\epsilon \frac{1}{1-c(j)K_\epsilon} b(j)\right) \end{aligned} \tag{5.123}$$

Indeed, it is easy to check that

$$\det \mathcal{B} = S^{(2)D}_{BB-BB}(j_a, j_r) S^{(2)D}_{BB-BB}(0,0). \tag{5.124}$$

Hence, from this understanding of the different couplings in the quadratic action $S^{(2)D}$, one can deduce that

$$\xi^{(2)D}_{[\alpha]} = \frac{\det \mathcal{B}_{\mathrm{I}}}{\det \mathcal{B}} = \frac{S^{(2)D}_{BB-BB}(j_a,0) S^{(2)D}_{BB-BB}(0,j_r)}{S^{(2)D}_{BB-BB}(0,0) S^{(2)D}_{BB-BB}(j_a,j_r)}. \tag{5.125}$$

The factorization of $\xi^{(2)D}_{[\alpha]}$ in the lemma then follows from using the formula (5.123) in the four factors of (5.125).

□

The definitions (5.119), (5.120) and (5.121) of the generating functions show that the next property holds.

Property 5.6 *Suppose the graph simple. For any j_a and j_r in a sufficiently small neighborhood of the origin, and for any $o \in \{D, C\}$ and any $x \in \{dd, \dot{o}o, do\}$,*

$$\xi^{(2)o}_{[\alpha],(x)}(j_a, 0) = \xi^{(2)o}_{[\alpha],(x)}(0, j_r) = 1.$$

In particular, the same is true for their products $\xi^{(2)\circ}_{[\alpha]}$ *and for the full second order generating function* $\xi^{(2)}_{[\alpha]}$.

5.3.2 Gaussian Autocorrelation Functions

The derivatives have now to be taken on the second order generating function $\xi^{(2)}_{[\alpha]}(j_a, j_r)$ given in Lemma 5.5, and the contribution $\xi^{\mathrm{MF}(2)}_{[\alpha]}(j_a, j_r)$ of the mean field has to be removed. The zero mode is diagonal in direction space, and hence, the contribution of the mean field to the full second order generating function $\xi^{(2)}_{[\alpha]}(j_a, j_r)$ is still given by the formula (5.54) obtained in the preceding section. Hence, if κ is the parameter equal to 1 if time-reversal symmetry is broken and 2 otherwise, the Gaussian generating function reads

$$\xi^{G}_{[\alpha]} = \prod_{\substack{x\in\{dd,\\ oo,do\}}} \xi^{G,D}_{[\alpha],(x)} \prod_{\substack{x\in\{dd,\\ oo,do\}}} \left(\xi^{G,C}_{[\alpha],(x)}\right)^{\kappa-1}, \tag{5.126}$$

with

$$\xi^{G,\circ}_{[\alpha],(dd)} \equiv \frac{\xi^{(2)\circ}_{[\alpha],(dd)}}{\xi^{\mathrm{MF}(2)\circ}_{[\alpha]}}, \quad \xi^{G,\circ}_{[\alpha],(oo)} \equiv \xi^{(2)\circ}_{[\alpha],(oo)}, \quad \xi^{G,\circ}_{[\alpha],(do)} \equiv \xi^{(2)\circ}_{[\alpha],(do)}. \tag{5.127}$$

for $\circ \in \{D, C\}$. The cooperon quantities exist only for $\kappa = 2$.

Lemma 5.7 *The derivatives of the Gaussian generating function of a simple graph satisfy*

$$\delta\xi^{G}_{[\alpha]} = \sum_{x\in\{dd,oo,do\}} \delta\xi^{G,D}_{[\alpha],(x)} + (\kappa-1) \sum_{x\in\{dd,oo,do\}} \delta\xi^{G,C}_{[\alpha],(x)}.$$

Moreover, if the graph is ergodic, then $\epsilon^{q-1}\xi^{G,D}_{[\alpha],(oo)}$ *and* $\epsilon^{q-1}\xi^{G,C}_{[\alpha],(oo)}$ *are both of order* $\mathcal{O}(\epsilon)$ *for any* $q \geq 2$.

Proof. By the properties 5.2 and 5.6, taking the advanced derivative on $\xi^{G}_{[\alpha]}$ written as in (5.126), and evaluating the resulting expression at $j_a = 0$, leads to

$$\frac{\partial}{\partial j_a}\xi^{G}_{[\alpha]}(0, j_r) = \frac{\partial}{\partial j_a}\xi^{G,D}_{[\alpha]}(0, j_r) + (\kappa-1)\frac{\partial}{\partial j_a}\xi^{G,C}_{[\alpha]}(0, j_r), \tag{5.128}$$

and

$$\frac{\partial}{\partial j_a}\xi^{G,\circ}_{[\alpha]}(0,j_r) = \frac{\partial}{\partial j_a}\xi^{G,\circ}_{[\alpha],(dd)}(0,j_r) + \frac{\partial}{\partial j_a}\xi^{(2)\circ}_{[\alpha],(oo)}(0,j_r) + \frac{\partial}{\partial j_a}\xi^{(2)\circ}_{[\alpha],(do)}(0,j_r), \tag{5.129}$$

for $\circ \in \{D, C\}$. Therefore, $\partial_a \xi^G_{[\alpha]}(0, j_r)$ can be seen as the sum of six terms, and the decomposition in the lemma follows. Moreover, if the graph is ergodic, Lemma 5.3 ensures that K has no eigenvalue 1. Therefore, taking retarded derivatives on a (oo) generating function in (5.129) does not create any singularity at all since, as it can be seen on (5.121), the limits $\epsilon \to 0$ of these functions exist and are analytic in a neighborhood of the origin. This finishes the proof of the lemma.

□

We are now ready to calculate the expression of the Gaussian autocorrelation functions $C^G_{[\alpha]}$ defined in (5.10). We first consider the case where the source supermatrices $J_a(j_a)$ and $J_r(j_r)$ are defined by (3.1)-(3.3). The corresponding result is given in the next theorem.

Theorem 5.8 *Suppose the graph simple and ergodic. Then, the Gaussian autocorrelation functions read*

$$C^G_{[\alpha]} = C^{G,D}_{[\alpha]} + (\kappa - 1)C^{G,C}_{[\alpha]},$$

where

$$C^{G,D}_{[\alpha]} = C^{G,D}_{[\alpha],(dd)} = \frac{1}{(q-1)(2B)^q}\sum_{\substack{k,l=0\\k\neq l}}^{q-1} R_{\alpha_k\alpha_l} - \frac{q}{(2B)^{q+1}},$$

and $C^{G,C}_{[\alpha]}$ is obtained from this formula by substituting $\hat{\alpha}_0$ for α_0.

Proof. In order to prove this theorem, it is sufficient to apply Lemma 5.7, use the result (5.82) found in the preceding section, and show that

$$\xi^{(2)\circ}_{[\alpha],(do)}(j) = 1, \quad \text{and hence,} \quad \delta\xi^{G,\circ}_{[\alpha],(do)} = 0. \tag{5.130}$$

These identities follow from the expression (5.121) for the (do) generating function and the fact that the functions a° and b° defined in (5.92) and (5.95) are always zero if the source supermatrices are defined as in (3.1)-(3.3).

□

Consider now two directed bonds $\alpha, \alpha' \in \mathbb{N}_{2B}$, and let $J_a^\times(j_a)$ and $J_r^\times(j_r)$ be the corresponding crossed source supermatrices, that is

$$J_a^\times(j_0) \equiv 1 + j_0 E_B \otimes E^{\alpha,\alpha'} \quad J_r^\times(j_1) \equiv 1 + j_1 E_B \otimes E^{\alpha',\alpha}, \tag{5.131}$$

as in (4.95). The crossed generating function $\xi^\times_{[\alpha,\alpha']}(j_a, j_r)$ defined in (4.96) from $J_a^\times(j_a)$ and $J_r^\times(j_r)$ has a second order approximation $\xi^{\times(2)}_{[\alpha]}(j_a, j_r)$, which induces a Gaussian autocorrelation function $C^{\times,G}_{[\alpha,\alpha']}$ defined from $\xi^{\times(2)}_{[\alpha,\alpha']}(j_a, j_r)$ by the formula (5.10). Notice that, apart from Theorem 5.8, in which the writing (3.1) and (3.2) of the source supermatrices is explicitly used, the results exposed in this section apply to this situation. The result is as follows.

Theorem 5.9 *Suppose the graph simple and ergodic. Then, the crossed Gaussian autocorrelation functions read*

$$\begin{aligned} C^{\times G}_{[\alpha,\alpha']} &= C^{\times G,D}_{[\alpha,\alpha']} + (\kappa-1) C^{\times G,C}_{[\alpha,\alpha']} \\ &= \frac{\delta_{\alpha,\alpha'}}{(2B)^2}\Big[2R_{\alpha\alpha} + (\kappa-1)\big(R_{\hat\alpha\alpha} + R_{\alpha\hat\alpha}\big)\Big] - \kappa\frac{2\delta_{\alpha,\alpha'}}{(2B)^3} \\ &\quad + \kappa\frac{\delta_{\alpha,\hat\alpha'}}{(2B)^2}\Big[R^K_{\alpha\alpha} + R^K_{\hat\alpha\hat\alpha}\Big] \end{aligned}$$

where the back-scattering matrix R^K *is defined by*

$$R^K_{\alpha\alpha'} \equiv \left(\frac{K^2}{1-K^2}\right)_{\alpha\alpha'} = \delta_{\alpha,\alpha'}\frac{M_{\alpha\hat\alpha}M_{\hat\alpha\alpha}}{1 - M_{\alpha\hat\alpha}M_{\hat\alpha\alpha}}.$$

Proof. By Lemma 5.7,

$$C^{\times G}_{[\alpha,\alpha']} = C^{\times G}_{[\alpha,\alpha'],(dd)} + C^{\times G}_{[\alpha,\alpha'],(do)}. \tag{5.132}$$

The first term is given by the result (5.74) of the preceding section. It occupies the first line of the right-hand side in the theorem. Besides, Lemma 5.7 claims

that the second term of (5.132) factorizes as a diffusion component, and a cooperon component if $\kappa = 2$. In order to compute this (do) autocorrelation function, the two derivatives on the (do) generating function (5.121) have to be taken. By a calculation similar to the one leading to (5.44), one gets

$$\delta\xi^{(2)\circ}_{[\alpha,\alpha'],(do)} = \mathrm{tr}\left[\frac{1}{1-N^\circ_\epsilon}N^\circ_{\epsilon,0}\frac{1}{1-N^\circ_\epsilon}N^\circ_{\epsilon,1} + \frac{1}{1-N^\circ_\epsilon}N^\circ_{\epsilon,01}\right], \tag{5.133}$$

where the function $N^\circ_\epsilon(j_0, j_1)$ is defined in (5.122), N°_ϵ denotes its value at the origin, and $N^\circ_{\epsilon,0}$, $N^\circ_{\epsilon,1}$ and $N^\circ_{\epsilon,01}$ stand for its derivatives at the origin. Since the functions a° and b° in (5.92) and (5.95) are both zero at the origin, N_ϵ, $N_{\epsilon,0}$ and $N_{\epsilon,1}$ all vanish, and hence, the first term in the trace of (5.133) vanishes as well. Therefore, (5.133) becomes

$$\delta\xi^{(2)\circ}_{[\alpha,\alpha'],(do)} = \mathrm{tr}\left[\left(\frac{M_\epsilon}{1-M_\epsilon}\right)\left(a^\circ_0\frac{K}{1-K}b^\circ_1 + a^\circ_1\frac{K}{1-K}b^\circ_0\right)\right]. \tag{5.134}$$

The functions a° and b° can be specialized to the crossed convention (5.131). One finds

$$\begin{aligned} a^D(j_0,j_1)_{\beta\beta'} &= a^C(j_0,j_1)_{\beta\beta'} = j_0\delta_{\beta,\beta'}\delta_{\alpha'\beta}\delta_{\alpha'\dot{\alpha}} + j_1\delta_{\dot{\beta}',\beta}\delta_{\alpha'\beta}\delta_{\alpha'\dot{\alpha}} \\ b^D(j_0,j_1)_{\beta\beta'} &= b^C(j_0,j_1)_{\beta\beta'} = j_0\delta_{\beta,\beta'}\delta_{\alpha\beta}\delta_{\alpha\alpha'} + j_1\delta_{\dot{\beta}',\beta}\delta_{\alpha'\beta}\delta_{\alpha'\dot{\alpha}} \end{aligned} \tag{5.135}$$

The corresponding (do) Gaussian autocorrelation functions are then given by the formula (5.20). They read

$$C^{\times G}_{[\alpha,\alpha'],(do)} = \kappa\lim_{\epsilon\to 0}\frac{\epsilon}{B}\delta_{\alpha'\dot{\alpha}}\left(\frac{M_\epsilon}{1-M_\epsilon}\right)_{\alpha\dot{\alpha}}\left[\left(\frac{K}{1-K}\right)_{\alpha\alpha} + \left(\frac{K}{1-K}\right)_{\dot{\alpha}\dot{\alpha}}\right] \tag{5.136}$$

Then, by Corollary 1.2, this limit becomes

$$C^{\times G}_{[\alpha,\alpha'],(do)} = \kappa\frac{\delta_{\alpha'\dot{\alpha}}}{(2B)^2}\left[\left(\frac{K}{1-K}\right)_{\alpha\alpha} + \left(\frac{K}{1-K}\right)_{\dot{\alpha}\dot{\alpha}}\right]. \tag{5.137}$$

Finally, observe that the diagonal elements of K^n vanish if n is odd, and apply Lemma 5.3

□

The fact that the Gaussian autocorrelation functions $C^G_{[\alpha,\alpha']}$ depend on the convention chosen for the source supermatrices shows that our Gaussian expansion scheme is not a second order expansion in any intrinsic parameter of the quantum graph.

Chapter 6

Criteria for Universality

6.1 Weak Conjectures

6.1.1 Gaussian Autocorrelations and Universal Models

The calculation scheme summarized in (5.9) leads to truncated autocorrelation functions, which are sums of mean field contributions obtained in Chapter 4 and Gaussian contributions obtained in Chapter 5. In this section, the Gaussian quantities are considered to be those given in Theorem 5.8. The importance of these truncated autocorrelation functions is twofold. Firstly, their Gaussian contributions, which are system-dependent, can be compared with their universal mean field parts in the limit of large graphs. These comparisons lead to conditions on the increasing sequence of quantum graphs $\{(G_l, S_l)\}_{l\in\mathbb{N}}$ to asymptotically follow the universal Gaussian models introduced in Section 2.2. Secondly, for a class of increasing sequences larger than this universal class, the truncated quantities approximate the exact autocorrelation functions.

The truncated moments $\tilde{M}_\alpha(q)$ are obtained by summing the mean field

contribution (4.75) and the Gaussian moments in Theorem 5.8. They read

$$\tilde{M}_\alpha(q) = \begin{cases} \frac{q!}{(2B)^q} + \frac{q}{(2B)^q} R_{\alpha\alpha} - \frac{q}{(2B)^{q+1}}, & \kappa = 1 \\ \frac{q!}{(2B)^q} + \frac{1}{(2B)^q} \left[R_{\dot{\alpha}\alpha} + R_{\alpha\dot{\alpha}} + 2(q-1) R_{\alpha\alpha} \right] - \frac{2q}{(2B)^{q+1}}, & \kappa = 2 \end{cases} \tag{6.1}$$

Similarly, if α and α' are two directed bonds, the corresponding approximated autocorrelation reads

$$\tilde{C}_{[\alpha,\alpha']} = \begin{cases} \frac{1+\delta_{\alpha,\alpha'}}{(2B)^2} + \frac{1}{(2B)^2} \left[R_{\alpha\alpha'} + R_{\alpha'\alpha} \right] - \frac{2}{(2B)^3}, & \kappa = 1 \\ \frac{1+\delta_{\alpha,\alpha'}+\delta_{\alpha,\dot{\alpha}'}}{(2B)^2} + \frac{1}{(2B)^2} \left[R_{\alpha\alpha'} + R_{\alpha'\alpha} + R_{\dot{\alpha}\alpha'} + R_{\alpha'\dot{\alpha}} \right] - \frac{4}{(2B)^3}, & \kappa = 2 \end{cases} \tag{6.2}$$

The expression for more general autocorrelation functions can also easily be found. Suppose that the elements in the vector $[\alpha] = [\alpha_0, \ldots, \alpha_{q-1}]$ are in the set $\{\beta_1, \ldots, \beta_n\}$, and that β_j appears $q_j \in \mathbb{N}$ times in $[\alpha]$ for $1 \leq j \leq n$. Then, for $\kappa = 1$,

$$\tilde{C}_{[\alpha]} = \frac{\prod_{j=1}^n q_j!}{(2B)^q} + \frac{1}{(q-1)(2B)^q} \sum_{\substack{k,l=0 \\ k \neq l}}^{q-1} R_{\alpha_k \alpha_l} - \frac{q}{(2B)^{q+1}}. \tag{6.3}$$

Suppose now that the bonds supporting the directed bonds in $[\alpha]$ form the set $\{b_1, \ldots, b_n\}$ and that each b_j, $1 \leq j \leq n$, supports $q'_j \in \mathbb{N}$ elements. Then, for $\kappa = 2$,

$$\begin{aligned} \tilde{C}_{[\alpha]} \;=\; & \frac{\prod_{j=1}^n q'_j!}{(2B)^q} + \frac{1}{(q-1)(2B)^q} \Bigg[\sum_{k=1}^{q-1} \Big(R_{\alpha_0 \alpha_k} + R_{\alpha_k \alpha_0} \\ & + R_{\dot{\alpha}_0 \alpha_k} + R_{\alpha_k \dot{\alpha}_0} \Big) + 2 \sum_{\substack{k,l=1 \\ k \neq l}}^{q-1} R_{\alpha_k \alpha_l} \Bigg] - \frac{q}{(2B)^{q+1}}. \end{aligned} \tag{6.4}$$

All these formulae asymptotically reduce to their mean field terms if and only if $R \to 0$ as $B \to \infty$. This motivates the following conjecture.

Conjecture 6.1 *An increasing sequence $\{(G_l, S_l)\}_{l \in \mathbb{N}}$ of ergodic simple graphs asymptotically follows the universal Gaussian random waves model if and only if the sequence of matrices $\{R_l\}_{l \in \mathbb{N}}$ converges to zero, that is, if and only if all the non-zero masses in Corollary 1.2 tend to one.*

The rest of this subsection is devoted to empirically justify why the truncated quantities determine the behavior of the exact quantities. In Theorem 5.8, it is shown that only the modes that are diagonal in direction space can contribute, and hence, the full second order action is given by $S^{(2)D}_{(dd)}$ in (5.36) and (5.37) if $\kappa = 1$, and by an additional cooperon action if $\kappa = 2$. The result of the Gaussian superintegral, which is given in (5.38) and (5.40), shows that the quadratic action can essentially be regarded as

$$\begin{pmatrix} v_1^\dagger & v_2^\dagger \end{pmatrix} \begin{pmatrix} \mathbb{1}_{\mathcal{A}} - s^\circ(j_a, j_r) M_\epsilon & 0 \\ 0 & \mathbb{1}_{\mathcal{A}} - M_\epsilon \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{6.5}$$

for the part involving the commuting variables, and

$$\begin{pmatrix} \chi_1^\dagger & \chi_2^\dagger \end{pmatrix} \begin{pmatrix} \mathbb{1}_{\mathcal{A}} - s^\circ(j_a, 0) M_\epsilon & 0 \\ 0 & \mathbb{1}_{\mathcal{A}} - s^\circ(0, j_r) M_\epsilon \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \tag{6.6}$$

for the part involving the anticommuting variable. Here the fields v_1, v_2, χ_1^*, χ_2^*, χ_1 and χ_2 are some combinations of the fields in (5.36) and (5.37), and the functions s°, defined in (5.33) and (5.39), are such that $s^\circ(0,0) = 1$. Each block in (6.5) and (6.6) can now be diagonalized. This diagonalization procedure provides new fields, say w_1, w_2, $\varphi_1^\dagger$, $\varphi_2^\dagger$, φ_1 and φ_2, and each of their components is associated with an eigenvalue m_j of $\mathbb{1}_{\mathcal{A}} - M$. For example, the $v_1^\dagger - v_1$ coupling in (6.5) becomes

$$\begin{pmatrix} w_1^*(1) & \dots & w_1^*(2B) \end{pmatrix} \begin{pmatrix} m_{\epsilon,1}(j_a, j_r) & & \\ & \ddots & \\ & & m_{\epsilon,2B}(j_a, j_r) \end{pmatrix} \begin{pmatrix} w_1(1) \\ \vdots \\ w_1(2B) \end{pmatrix}, \tag{6.7}$$

where the numbers $m_i \equiv m_{0,i}(0,0)$ are the masses introduced in Corollary 1.2, that is, the eigenvalues of $\mathbb{1}_{\mathcal{A}} - M$. The other blocks in (6.5) and (6.6) generate similar expressions. One can suppose that $m_1 = 0$. In this case, $w_1(1)$, $w_2(1)$, $\varphi_1^*(1)$, $\varphi_2^*(1)$, $\varphi_1(1)$ and $\varphi_2(1)$ exactly correspond to the components Y^T_{BB}, Y^T_{FF}, $\tilde{Y}_{BF}$, $\tilde{Y}_{FB}$, Y_{FB} and Y_{BF} of the zero mode Y. In fact, the association between zero mode and uniform component of $\mathbb{1}_{\mathcal{A}} - M$ has already been noticed after

(5.66) by comparing the results of the second order and mean field second order superintegrals. The other components of the fields correspond to non-zero masses, and can thus be called *massive modes.*

The expansions of the three logarithms in the exact action in Corollary 3.5 as Taylor series lead to

$$S = S^{(2)} + S^{(4)} + S^{(6)} + \dots, \tag{6.8}$$

where S^{2n} is of order $2n$ in the fields Z and $\tilde{Z}$. The expansion (6.8) in turns yields a series

$$C_{[\alpha]} = C^{\mathrm{MF}}_{[\alpha]} + C^{(2)}_{[\alpha]} + C^{(4)}_{[\alpha]} + C^{(6)}_{[\alpha]} + \dots \tag{6.9}$$

for the exact autocorrelation functions, where each term of $C^{2n}_{[\alpha]}$ contains $2n-1$ components of R, that is $2n-1$ inverse non-zero masses. Within this new notation, the truncated autocorrelation functions (6.3) and (6.4) read

$$\tilde{C}_{[\alpha]} = C^{\mathrm{MF}}_{[\alpha]} + C^{(2)}_{[\alpha]}. \tag{6.10}$$

We first define the series (6.9) and then give an argument concerning $C^{(4)}_{[\alpha]}$ in the case of a single retarded source j_r. The same idea can then easily be applied to the general case. From (6.8), the exact generating function can be written

$$\xi_{[\alpha]}(j) = \int d(w_1, w_2, \varphi_1, \varphi_2) e^{-S^{(2)}(j)} \left[1 + c^{(4)}(j) + c^{(6)}(j) + \dots\right] \tag{6.11}$$

where $c^{(4)}$ is quartic in the fields, $c^{(6)}$ is of order six, and so on. In (6.11), the polynomials $c^{(4)}, c^{(6)}, \dots$ come from $S - S^{(2)}$ and not from expanding the exponential of $S^{(2)}$. The first term in the square brackets of (6.11) is the second order part investigated in Chapter 5. It provides a divergent mean field contribution, and an autocorrelation function involving one component of R in each term. Now, by (6.7) and by Wick Theorem, a term in $c^{(4)}$ provides two factors $m_{\epsilon,i}(j_a, j_r)^{-1}$. The derivatives thus yield four factors $\frac{1-m_{\epsilon,i}}{m_{\epsilon,i}}$. By Theorem 3.1, these derivatives must be multiplied by ϵ and the limit $\epsilon \to 0$ must then be taken, so that a term in which each factor $\frac{1-m_{\epsilon,i}}{m_{\epsilon,i}}$ is chosen with a

non-zero mass cannot contribute. On the opposite extreme, if the four factors $\frac{1-m_{\epsilon,i}}{m_{\epsilon,i}}$ are chosen with the zero mass $m_{\epsilon,1}$, the resulting term behaves like ϵ^{-4}, which is too singular. Notice that the divergent contribution in Chapter 5 also comes from choosing all the masses to be $m_{\epsilon,1}$. One deduces that the contributions in (6.11) containing the zero mode only diverge order by order. However, in Chapter 4, the superintegral (6.11) restricted to the zero mode is exactly performed and a finite result is obtained. Therefore, cancellations have to happen in the series (6.11) restricted to the zero mode. If now, among the four factors $\frac{1-m_{\epsilon,i}}{m_{\epsilon,i}}$, three or two factors are chosen with the zero mass $m_{\epsilon,1}$, the behavior is in ϵ^{-3} or in ϵ^{-2}, which is still too singular. Since, after all, the exact autocorrelation functions $C_{[\alpha]}$ are well-defined, one expects these divergencies to be compensated by other divergencies in the higher order terms in (6.11), just as in the zero mode series. The relevant terms contain one factor $\frac{1-m_{\epsilon,1}}{m_{\epsilon,1}}$ and three massive factors $\frac{1-m_i}{m_i}$, that is three factors R. They form the contribution $C^{(4)}_{[\alpha]}$ in (6.9). This argument indeed corroborates the general claim (6.8)-(6.9).

The claim (6.8)-(6.9) shows that, if all the masses m_i tend to one as $B \to \infty$, then the terms $C^{(2n)}_{[\alpha]}$ in (6.9) all vanish as $B \to \infty$, and hence $C_{[\alpha]} = C^{\mathrm{MF}}_{[\alpha]}$. This conclusion leads to the conjecture. Moreover, this claim also implies that, if all the masses lie between $\frac{1}{2}$ and 2, so that $\frac{m_i}{1-m_i}$ is less than unity, the second order is the leading term in the series (6.9), and hence, the truncated formulae (6.3) and (6.4) can be expected to approximate the exact quantities. In fact, one can still expect the truncated autocorrelations to approximate the exact autocorrelations if some masses do asymptotically vanish, provided they do not do so too fast.

6.1.2 Gaussian Autocorrelations and Fluctuations

The truncated autocorrelation functions (6.2), together with the formula (2.22), generate a truncated expression $\tilde{\mathcal{F}}_V$ for the fluctuations of an observable V.

The asymptotic quantum ergodicity issue described in Section 2.1 can be discussed in terms of these truncated fluctuations. Moreover, in the situations where $\tilde{\mathcal{F}}_V$ decays, $\tilde{\mathcal{F}}_V$ is expected to approximate the exact fluctuations $\mathcal{F}_V$.

A direct calculation shows that, for an observable V with $\bar{V} = 0$,

$$\tilde{\mathcal{F}}_V = \kappa \frac{\mathrm{tr}(VL)^2}{(\mathrm{tr}L)^2} + 2\kappa \frac{\sum_{\beta,\beta'} [VL \cdot R \cdot VL]_{\beta\beta'}}{(\mathrm{tr}L)^2} \tag{6.12}$$

This formula motivates the following criterion for asymptotic quantum ergodicity to be met in an increasing sequence of quantum graphs.

Conjecture 6.2 *An increasing sequence $\{(G_l, S_l)\}_{l\in\mathbb{N}}$ of ergodic simple graphs is asymptotically quantum ergodic if and only if*

$$\lim_{l\to\infty} \frac{\sum_{\beta,\beta'} [V_l L_l \cdot R_l \cdot V_l L_l]_{\beta\beta'}}{(trL_l)^2} = 0$$

for any acceptable sequence $\{V_l\}_{l\in\mathbb{N}}$ with $\bar{V}_l = 0$.

This conjecture relies on two facts. Firstly, suppose that the second order term $C^{(2)}_{[\alpha]}$ in (6.9) does not asymptotically vanish. Since the other terms $C^{(2n)}_{[\alpha]}$ are of higher order in the inverse masses, they cannot compensate the non-vanishing term $C^{(2)}_{[\alpha]}$, and hence, the fluctuations (6.12) do not decay. Secondly, in order for the truncated fluctuations to decay, the non-zero masses m_i, $2 \leq i \leq 2B$, have either to stay away from zero, or to approach zero slowly enough. This statement will be made more precise in Section 6.2. In such a slow approach situation, the evolution of the exact autocorrelations as B increases is captured by the lowest order term in the inverse masses, that is, by the truncated fluctuations.

Moreover, if the stronger condition $R_l \to 0$ is fulfilled, Conjecture 6.1 states that full universality is met. The convergence rate of $\tilde{\mathcal{F}}_V$, and hence of $\mathcal{F}_V$, is then universal.

6.2 Strong Conjectures

6.2.1 Universal Models and Fluctuations

The crossed formulae for the Gaussian autocorrelation functions in Theorem 5.9 differ from the formulae in Theorem 5.8, which are used in the previous section. These expressions only involve the diagonal components (α, α) and the components $(\alpha, \hat{\alpha})$ of the matrix R. However, they also contain an extra back-scattering term. In this section, the universal Gaussian models and the asymptotic quantum ergodicity issues are discussed in terms of the crossed formulae.

With the Gaussian components given by Theorem 5.9 instead of Theorem 5.8, the formula (6.2) for the autocorrelations $\tilde{C}_{[\alpha,\alpha']}$ of degree two becomes

$$\begin{aligned} C^{\times}_{[\alpha,\alpha']} &= \frac{1+\delta_{\alpha,\alpha'}+(\kappa-1)\delta_{\alpha,\hat{\alpha}'}}{(2B)^2}+\frac{\delta_{\alpha,\alpha'}}{(2B)^2}\Big[2R_{\alpha\alpha}+(\kappa-1)\left(R_{\hat{\alpha}\alpha}+R_{\alpha\hat{\alpha}}\right)\Big] \\ &\quad -\kappa\frac{\delta_{\alpha,\alpha'}}{(2B)^3}+\kappa\frac{\delta_{\alpha,\hat{\alpha}'}}{(2B)^2}\Big[R^K_{\alpha\alpha}+R^K_{\hat{\alpha}\hat{\alpha}}\Big] \end{aligned} \tag{6.13}$$

It is not difficult to see that, in (6.13), the Gaussian contribution from the diagonal modes in direction space, namely the second and third terms, are obtained from the Gaussian contributions in (6.2) by inserting a factor $\delta_{\alpha,\alpha'}$. This remark is actually already made in (5.74). Besides, the back-scattering term of $C^{\times}_{[\alpha,\alpha']}$, namely the last term in (6.13) vanishes if $\alpha = \alpha'$, and hence, for $\alpha = \alpha'$, the formulae (6.2) and (6.13) agree.

Despite their different aspects, the truncated autocorrelations in (6.2) and (6.13) are obtained from the same calculation scheme, starting with two equivalent initial formulae, and can thus be expected to incorporate the same dynamical characteristics of quantum graphs. The next conjecture is the analog of Conjecture 6.1 obtained from the truncated autocorrelations (6.13).

Conjecture 6.3 *An increasing sequence $\{(G_l, S_l)\}_{l\in\mathbb{N}}$ of ergodic simple graphs asymptotically follows the universal Gaussian random waves model if and only if it satisfies*

- $R_{l,\alpha\alpha}, R^K_{l,\alpha\alpha} \to 0$ *as* $l \to \infty$ *for all* $\alpha \in \mathbb{N}_{2B}$, *if* $\kappa = 1$
- $R_{l,\alpha\alpha}, R_{l,\alpha\hat{\alpha}}, R^K_{l,\alpha\alpha} \to 0$ *for all* $\alpha \in \mathbb{N}_{2B}$, *if* $\kappa = 2$

The crossed truncated autocorrelations (6.13), together with (2.22), generate a formula $\tilde{\mathcal{F}}_V$ for the fluctuations of an observable V. If $\bar{V} = 0$, this formula reads

$$\begin{aligned}\tilde{\mathcal{F}}^{\times}_V &= \kappa\frac{\mathrm{tr}(VL)^2}{(\mathrm{tr}L)^2} + \frac{2\mathrm{tr}(VL)^2R}{(\mathrm{tr}L)^2} + (\kappa-1)\frac{2\mathrm{tr}(VL)^2R\sigma_1^d}{(\mathrm{tr}L)^2} \\ &\quad -\kappa\frac{\mathrm{tr}(VL)^2}{(2B)(\mathrm{tr}L)^2} + \kappa\frac{2\mathrm{tr}(VL)^2R^K}{(\mathrm{tr}L)^2}\end{aligned} \tag{6.14}$$

Let us now consider observables V such that $V_bL_b = \frac{\mathrm{tr}L}{2B}$ on half of the bonds, and $V_bL_b = -\frac{\mathrm{tr}L}{2B}$ on the other half. The set of such observables is actually sufficiently large to compare the intensities of the wavefunction on the different bonds. Moreover, they provide acceptable sequences, according to (2.6). For such observables,

$$\begin{aligned}\tilde{\mathcal{F}}^{\times}_V &= \kappa\frac{1}{2B} + \frac{2\mathrm{tr}R}{(2B)^2} + (\kappa-1)\frac{2\mathrm{tr}R\sigma_1^d}{(2B)^2} \\ &\quad -\kappa\frac{1}{(2B)^2} + \kappa\frac{2\mathrm{tr}R^K}{(2B)^2}.\end{aligned} \tag{6.15}$$

It will be seen in 6.2.2 that the back-scattering term, namely the fifth term of (6.15) decays if the massive term, that is the second term, decays in the limit of large graphs, and hence, it cannot forbid asymptotic quantum ergodicity on its own. Besides, since this back-scattering term is positive, it cannot compensate a possible residual massive contribution to (6.15) in the limit of large graphs. These remarks motivate the next conjecture.

Conjecture 6.4 *An increasing sequence* $\{(G_l, S_l)\}_{l\in\mathbb{N}}$ *of ergodic simple graphs with* $\kappa = 1$ *is asymptotically quantum ergodic if and only if*

$$\lim_{l\to\infty}\frac{trR_l}{(2B)^2} = \lim_{l\to\infty}\frac{1}{(2B_l)^2}\sum_{i=2}^{2B_l}\frac{1-m_{l,i}}{m_{l,i}} = 0.$$

If $\kappa = 2$, *such a sequence is asymptotically quantum ergodic if furthermore*

$$\lim_{l\to\infty}\frac{trR_l\sigma_1^d}{(2B)^2} = 0.$$

6.2.2 Bounds on Back-Scattering Terms

By Lemma 5.3, the eigenvalues of K^2 can be written $1 - \rho_\beta$, where

$$\rho_\beta = 1 - M_{\beta\hat{\beta}} M_{\hat{\beta}\beta}. \tag{6.16}$$

For a dynamically connected graph, $\rho_\beta \in (0, 2)$ for all $\beta \in \mathbb{N}$. Since $\rho_\beta = \rho_{\hat{\beta}}$, this eigenvalue is also denoted by ρ_b, where b is the bond supporting β and $\hat{\beta}$. Lemma 5.3 also asserts that $\rho_b = \rho_\beta \to 0^+$ as $B \to \infty$ if and only if the bond b becomes asymptotically disconnected, that is

$$x_\beta \equiv 1 - M_{\beta\hat{\beta}} \quad \text{and} \quad x_{\hat{\beta}} \equiv 1 - M_{\hat{\beta}\beta} \tag{6.17}$$

both asymptotically vanish. Let M_0 be a map obtained from M by completely disconnecting the bond b, and suppose that $\|M - M_0\|$ is small. First, from the trivial identity

$$\rho_b = \rho_\beta = x_\beta + x_{\hat{\beta}} - x_\beta x_{\hat{\beta}}, \tag{6.18}$$

one deduces that, at first order in the perturbation $\|M - M_0\|$, $\rho_b \sim x_\beta + x_{\hat{\beta}}$. Besides, M_0 has a second eigenvalue one, and one finds

$$M_0 \frac{1}{\sqrt{2}} \Big(|e_\beta\rangle + |e_{\hat{\beta}}\rangle \Big) = \frac{1}{\sqrt{2}} \Big(|e_\beta\rangle + |e_{\hat{\beta}}\rangle \Big). \tag{6.19}$$

One can imagine a continuous way to reach M from M_0, and this process transforms the eigenvalue $\lambda_b^0 \equiv 1$ of M_0 into an eigenvalue λ_b of M. At first order,

$$\begin{aligned} \lambda_b &\sim 1 + \frac{1}{2} \Big(\langle e_\beta| + \langle e_{\hat{\beta}}| \Big) \Big(M - M_0 \Big) \Big(|e_\beta\rangle + |e_{\hat{\beta}}\rangle \Big) \\ &\sim 1 + \frac{1}{2} \Big(M_{\beta\hat{\beta}} + M_{\hat{\beta}\beta} - M_{0\beta\hat{\beta}} - M_{0\hat{\beta}\beta} \Big) \\ &\sim \frac{1}{2} \Big(M_{\beta\hat{\beta}} + M_{\hat{\beta}\beta} \Big) \end{aligned} \tag{6.20}$$

The second expression in the right-hand side of (6.20) comes from the lack of loops on the graph. This short calculation shows that the mass $m_b \equiv 1 - \lambda_b$ behaves like

$$m_b \sim \frac{1}{2} \Big(x_\beta + x_{\hat{\beta}} \Big) \sim \frac{\rho_b}{2}. \tag{6.21}$$

In summary, with every bond b for which ρ_b tends to zero as $B \to \infty$, one can associate a mass m_b going to zero at the same rate.

The previous argument shows in particular that the spectral gap Δ_{K^2} of $\mathbb{1} - K^2$ cannot vanish faster than the spectral gap Δ_M of $\mathbb{1} - M$. In other terms, there is a constant $C \geq 0$ such that

$$\Delta_M \leq C\Delta_{K^2}. \tag{6.22}$$

We can now show that $\mathrm{tr}R_l$ controls $\mathrm{tr}R_l^K$ in the increasing sequence $\{(G_l, S_l)\}_{l\in\mathbb{N}}$. For convenience, we order the bonds $b \in \mathbb{N}_{B_l}$ in each graph of the sequence such that $\rho_{l,b} \leq \rho_{l,b+1}$ for all $b \in \mathbb{N}_{B_l-1}$. With this convention, the most disconnected bonds come first. Now, for each $l \in \mathbb{N}$, one introduces the natural number

$$h(l) \equiv \max\left\{b \in \mathbb{N}_{B_l} \middle| \lim_{l\to\infty} \rho_{l,b} = 0\right\} \in \mathbb{N}_{B_l}. \tag{6.23}$$

The increasing sequence $\{h(l)\}_{l\in\mathbb{N}}$ diverges if and only if infinitely many bonds get disconnected in the large graph limit $l \to \infty$. For any fixed $l \in \mathbb{N}$, the back-scattering contributions to the truncated fluctuations $\tilde{\mathcal{F}}^{\kappa}_{l,V_l}$ in (6.15) can be written

$$\tilde{\mathcal{F}}^{BS}_{l,V_l} \equiv \frac{4\kappa}{(2B_l)^2} \sum_{b=1}^{h(l)} \frac{1-\rho_{l,b}}{\rho_{l,b}} + \frac{4\kappa}{(2B_l)^2} \sum_{b=h(l)+1}^{B_l} \frac{1-\rho_{l,b}}{\rho_{l,b}}. \tag{6.24}$$

In the right-hand side, the factors 4κ come from the factor 2κ in the back-scattering term in (6.15) on one hand, and from the degeneracy $\rho_b \equiv \rho_\beta = \rho_{\bar{\beta}}$ of the spectrum of K^2 on the other hand. Notice that, by definition of $h(l)$, it exists $\delta > 0$ independent of l such that, for all $i \in \{h(l)+1, \ldots, B_l\}$, $\rho_{l,i} > \delta$. Hence, the second term of (6.24) can be bounded by

$$\begin{aligned}\left|\frac{4\kappa}{(2B_l)^2} \sum_{b=h(l)+1}^{B_l} \frac{1-\rho_{l,b}}{\rho_{l,b}}\right| &\leq \frac{4\kappa}{(2B_l)^2} \sum_{b=h(l)+1}^{B_l} \frac{1}{\delta} \\ &\leq \frac{4\kappa}{(2B_l)^2} \frac{B_l - h(l)}{\delta} \leq \frac{\kappa}{\delta B_l}.\end{aligned} \tag{6.25}$$

Therefore, the second term of (6.24) asymptotically vanishes, and one can write

$$\tilde{\mathcal{F}}^{BS}_{l,V_l} \sim \frac{4\kappa}{(2B_l)^2} \sum_{b=1}^{h(l)} \frac{1-\rho_{l,b}}{\rho_{l,b}}. \tag{6.26}$$

The meaning of this relation is that the left and right-hand sides either both go to zero, or both converge to some positive value, or both diverge as $l \to \infty$. Besides, for any fixed $l \in \mathbb{N}$, the massive contributions to the truncated fluctuations can also be split

$$\tilde{\mathcal{F}}^{M}_{l,V_l} \equiv \frac{2}{(2B_l)^2} \sum_{b=1}^{h(l)} \frac{1-m_{l,b}}{m_{l,b}} + \frac{2}{(2B_l)^2} \sum_{\substack{\text{other positive} \\ \text{masses } m_{l,i}}} \frac{1-m_{l,i}}{m_{l,i}}, \tag{6.27}$$

where, for $b \in \{1,\dots,h(l)\}$, $m_{l,b} \equiv 1-\lambda_{l,b}$ is the mass corresponding to $\rho_{l,b}$ in the construction (6.20)-(6.21). By (6.21) and (6.26), one can write for any fixed $l \in \mathbb{N}$

$$\tilde{\mathcal{F}}^{M}_{l,V_l} \sim \frac{1}{2\kappa}\tilde{\mathcal{F}}^{BS}_{l,V_l} + \frac{2}{(2B_l)^2} \sum_{\substack{\text{other positive} \\ \text{masses } m_{l,i}}} \frac{1-m_{l,i}}{m_{l,i}} \tag{6.28}$$

In the right-hand side of (6.27), the second term can behave in several ways as $l \to \infty$. However, it is always positive. Hence, $\tilde{\mathcal{F}}^{M}_{l,V_l} \to 0$ implies $\tilde{\mathcal{F}}^{BS}_{l,V_l} \to 0$.

6.2.3 Ergodicity and Classical Spectral Gap

Sufficient conditions for the first limit in Conjecture 6.4 to be zero or not can be given in terms of the sequence of spectral gaps $\{\Delta_{M_l}\}_{l\in\mathbb{N}}$ of $\mathbb{1}-M_l$. The spectrum of M is ordered such that $|m_i| \le |m_{i+1}|$ for all $i \in \mathbb{N}_{2B-1}$. With this ordering,

$$\Delta_M \equiv |m_2|. \tag{6.29}$$

Suppose first that the masses do not approach the origin, that is

$$\delta \equiv \inf_{l\in\mathbb{N}} \Delta_{M_l} > 0. \tag{6.30}$$

Then, clearly,

$$\tilde{\mathcal{F}}^{M}_{l,V_l} \equiv \frac{2}{(2B_l)^2} \sum_{i=2}^{2B_l} \frac{1-m_{l,i}}{m_{l,i}} \le \frac{2(2B_l-1)}{(2B_l)^2\delta} \tag{6.31}$$

which decays as $l \to \infty$.

Suppose now that the spectral gap approaches the origin, and that it does so with an exponential rate $\alpha > 0$, that is

$$|m_{l,2}| \sim (2B_l)^{-\alpha} \tag{6.32}$$

as $l \to \infty$. The matrix R has real coefficients since M is real, the vector $|1\rangle$ is real, and by definition

$$R \equiv \lim_{\epsilon \to 0} \left(\frac{M_\epsilon}{1 - M_\epsilon} - \frac{e^{-2\epsilon}}{1 - e^{-2\epsilon}} |1\rangle\langle 1| \right) \tag{6.33}$$

Therefore, the massive fluctuations can be written

$$\tilde{\mathcal{F}}_V^M \equiv \frac{2}{(2B)^2} \sum_{i=2}^{2B} \Re \frac{1 - m_i}{m_i} = \frac{2}{(2B)^2} \sum_{i=2}^{2B} \frac{\Re m_i}{|m_i|^2} - \frac{2(2B-1)}{(2B)^2}, \tag{6.34}$$

which shows that they decay if and only if

$$\hat{\mathcal{F}}_V^M \equiv \frac{2}{(2B)^2} \sum_{i=2}^{2B} \frac{\Re m_i}{|m_i|^2} \tag{6.35}$$

decays. The best possible estimates for the real part of a complex number m_i in the disc $|1 - z| \leq 1$ are

$$\Re m_i \leq |m_i| \quad \text{and} \quad |m_i|^2 \leq 2\Re m_i, \tag{6.36}$$

From the first inequality, we obtain

$$\hat{\mathcal{F}}_V^M \leq \frac{2}{(2B)^2} \sum_{i=2}^{2B} \frac{1}{|m_i|} \leq \frac{2}{(2B)^2} \frac{2B-1}{|m_2|} \sim (2B)^{\alpha-1}. \tag{6.37}$$

Therefore, if $\alpha < 1$, the massive fluctuations decay. The second inequality is not sharp enough to provide a sufficient criterion for the massive fluctuations not to decay. Indeed, it merely leads to

$$\hat{\mathcal{F}}_V^M > \frac{2}{(2B)^2} \frac{\Re m_2}{|m_2|^2} \geq \frac{1}{(2B)^2}, \tag{6.38}$$

which tends to zero irrespective of the spectral gap $|m_2|$. Notice that the second inequality in (6.36) is fulfilled as an equality if and only if m_i is on the

circle $|1-z|=1$. Therefore, the non-occurance of m_2 in the right-hand side of (6.38) comes from the possibility for the mass m_2 to approach the origin tangentially to the circle $|1-z|=1$. Suppose now that m_2 does not approach zero tangentially to this circle. In this case,

$$\gamma_2 \equiv \sup_{l\in\mathbb{N}} \left| \arg m_{l,2} \right| < \frac{\pi}{2}, \tag{6.39}$$

and the second inequality in (6.36) can be replaced with

$$|m_2| \leq (1+\tan\gamma_2)\cdot \Re m_2. \tag{6.40}$$

Making use of this new lower bound for the real part of m_2 yields

$$\hat{\mathcal{F}}_V^M > \frac{2}{(2B)^2}\frac{\Re m_2}{|m_2|^2} \geq \frac{2}{(2B)^2}\frac{1}{(1+\tan\gamma_2)|m_2|} \sim (2B)^{\alpha-2}. \tag{6.41}$$

This proves that, if (6.39) is fulfilled and $\alpha \geq 2$, the massive fluctuations do not decay.

Chapter 7

Discussions

Our main results are the formulae (6.3) and (6.4), which are expressions for the autocorrelation functions of the amplitudes $C_{[\alpha]}$ defined in (2.57). These formulae depend on the quantum graph only through the matrix R, and this matrix, which is defined in Corollary 1.2, only involves the underlying classical dynamics M. Hence, our results relate the statistical properties of the quantum energy eigenfunctions to properties of the classical dynamics on the graph. Moreover, they also reveal that the system dependency has no chance to vanish, and hence, a finite graph cannot be entirely described by the universal random waves model developed in Section 2.2 or even be quantum ergodic. These properties can only be met asymptotically in sequences of graphs in which the number of bonds diverges to infinity. The four conjectures listed in Chapter 6 provide classical criteria for such a sequence of growing quantum graphs to be asymptotically described by the universal random wave model or to be asymptotically quantum ergodic. The criteria for asymptotic full universality, that is for the universal model to be satisfied in the large graphs limit, are more restrictive than the criteria for asymptotic quantum ergodicity, since this latter property only depends on the second moment of the intensities, and the fluctuations $\tilde{\mathcal{F}}_V$ in (6.12), which measure the deviation to ergodicity, can possibly decay in a non-universal way as the number of bonds increases.

The general formulae (6.3) and (6.4) for the autocorrelation functions, and

in particular the formula (6.12) for the fluctuations $\tilde{\mathcal{F}}_V$, have been obtained by a saddle-point analysis of the exact field-theoretical expression in Theorem 3.4. A comparison with the two periodic orbits approaches in 2.1.4 and 3.2 reveals how the field-theoretical scheme exactly proceeds. The first term of $\tilde{\mathcal{F}}_V$ in (6.12), which is universal, originates from our exact calculation on the saddle-point manifold, and it coincides with the result predicted by the long diagonal orbits in 2.1.4. This draws a parallel between zero mode, that is uniform component of the classical map M, and long diagonal orbits. This is in fact not surprising since the zero mode is precisely the one that does not decay, and can thus survive in long orbits. The second term of $\tilde{\mathcal{F}}_V$ involves the system-dependent matrix R, that is the massive modes, and it coincides with the system-dependent contribution of the diagonal approximation exposed in 3.2. Hence, one deduces that our field-theoretical approach discriminates between the different modes of the classical map M. The uniform component of M is treated in an exact way, which the diagonal approximation in 3.2 cannot do, while the massive decaying modes are treated in a perturbative way.

It is also interesting to compare our results with those obtained by S. Gnutzmann and A. Altland in [40] and [41] concerning the asymptotic spectral two-point correlation function $R_2(s)$ in a sequence of growing quantum graphs. Their theory relates the function $R_2(s)$ to the sequence of spectral gaps Δ of the matrices $\mathbb{1} - M$. If the spectral gaps stay away from zero, the random matrix two-point correlation function is obtained in the limit of large graphs. The condition in Conjecture 6.1 for the universal random waves model to be met in this limit is that all the non-zero eigenvalues of $\mathbb{1} - M$ tend to one, which is obviously a much stronger requirement. Hence, even in situations where the random waves model does not hold, there is a possibility for random matrix theory to describe $R_2(s)$, but if the random waves model does hold, then $R_2(s)$ must be universal. Moreover, if the sequence of spectral gaps vanishes as $\Delta \sim B^{-\alpha}$ as the number of bonds B becomes large, Gnutzmann and Altland's theory predicts different outcomes for $R_2(s)$ depending on the

value of the positive number α. If $\alpha < \frac{1}{2}$, a random matrix behavior is reached, whereas a non-zero system-dependent contribution always remains if $\alpha \geq 1$. In the intermediate regime $\alpha \in [\frac{1}{2}, 1)$, the asymptotic two-point function $R_2(s)$ depends on the proportion of vanishing modes, as explained in [41]. In 6.2.3, we found that $\alpha < 1$ implies asymptotic quantum ergodicity, whereas $\alpha \geq 2$ forbids ergodicity. Therefore, universality for $R_2(s)$ implies asymptotic quantum ergodicity. However, in the domain $\alpha \in [\frac{1}{2}, 1)$, quantum ergodicity is always reached, whereas $R_2(s)$ can be non-universal.

To conclude, let us mention some possible improvements of our method and some interesting directions for further research. In the main formulae (6.3) and (6.4), the system-dependent terms originate from the Gaussian approximation around the point $Q = \sigma_3^{RA}$ in the directions that are transverse to the saddle-point manifold. A true Gaussian approximation should expand the exact action to second order around every point of the saddle-point manifold. The two procedures are indeed equivalent if the manifold parametrized by the general fields $(Z, \tilde{Z})$ looks the same around each saddle, but this fact should be proved. Moreover, in this second order expansion, the higher order terms have not been controlled. Estimating these terms remains a major problem of this field-theoretical method. Besides, it has been seen that different but equivalent conventions for the autocorrelation functions in Theorem 2.3 lead to different outcomes by our second order expansion scheme. In particular, the results provided by the parallel and crossed conventions, which serve as basis for the four conjectures in Chapter 6, do not agree in general. This proves that our second order expansion is not an expansion in any intrinsic parameter of the quantum graph. Another arising question is whether the formulae (6.3) and (6.4) are suitable to describe other quantum systems if the matrix M is replaced with the Perron-Frobenius operator. The field-theoretical method used here is probably difficult to generalize to other systems. An idea would be to develop a periodic orbit approach that reproduces the autocorrelation functions (6.3) and (6.4) and then to transfer it to other types of systems.

Appendix A

Time Inversion on Quantum Graphs

A.1 Time Inversion in Quantum Mechanics

A.1.1 Heuristic Introduction

The goal of this appendix is to briefly expose the general theory of time inversion in quantum mechanics, which can be found in the textbook [69] by Wigner, and to apply these general ideas to quantum graphs. Some notions about time inversion on quantum graphs, and in particular about the effects of this operation on a Bose-Fermi structure can be found in [41] and [44]. In A.2, the same notions are explained in more details, and the basic algebraic rules are developed.

Some isolated quantum systems admit the transformation consisting in reversing the direction of time $t \mapsto -t$ as a symmetry. In this case, there exists an operator θ acting on the Hilbert space of the physical states, which represents time inversion and commutes with the Hamiltonian. These systems are said to be *time-reversal invariant.* For such systems, the time inversion operator θ is required to satisfy the relation

$$U_t \cdot \theta \cdot U_t \cdot \theta = \lambda_1 \mathbb{1}, \tag{A.1}$$

where U_t is the evolution operator and $\lambda_1 \in \mathbb{C}$ is some constant. The physical meaning of θ implies that for any state ψ, $\theta^2\psi$ and the state ψ itself provide the same mean values. Hence, one must impose $\theta^2 = \lambda\mathbb{1}$, and equation (A.1) is thus equivalent to

$$U_t \cdot \theta = \lambda_2 \cdot \theta \cdot U_{-t} \tag{A.2}$$

for some $\lambda_2 \in \mathbb{C}$.

It is well-known that a symmetry is either described by a group of unitary operators or by an antiunitary operator. Indeed, these operators are precisely those preserving the transition probability $|(\psi_1, \psi_2)|$ between the states ψ_1 and ψ_2, that is

$$|(T\psi_1, T\psi_2)| = |(\psi_1, \psi_2)|. \tag{A.3}$$

Let us recall that an operator T is said to be antiunitary if and only if it satisfies (A.3) and it is antilinear, that is

$$T(c_1\psi_1 + c_2\psi_2) = c_1^* T(\psi_1) + c_2^* T(\psi_2) \tag{A.4}$$

for all $c_1, c_2 \in \mathbb{C}$ and all states ψ_1, ψ_2. By contrast, the unitary operators can be characterized as the linear operators satisfying (A.3).

Let us consider a time-reversal invariant system, that is a system having a symmetry θ satisfying (A.2), and let us show that θ cannot be unitary, and is thus antiunitary. In order to do that, suppose θ unitary, and assume for simplicity that the spectrum of the Hamiltonian is discrete. Then, let $\{\psi_k\}$ be a complete orthonormal set of eigenfunctions of the Hamiltonian. Since θ is a symmetry of the system, it must commute with the Hamiltonian, and $\theta\psi_k$ is also an eigenfunction of the Hamiltonian with same eigenvalue as ψ_k, say E_k. It follows that, for any $\phi = \sum_k \alpha_k\psi_k$, $\alpha_k \in \mathbb{C}$, , the formula

$$U_t\theta\phi = \sum_k \alpha_k e^{-iE_k t}\theta\psi_k \tag{A.5}$$

yields the time evolution of the state $\theta\phi$. On the other hand, since the operator θ is assumed unitary,

$$\theta U_{-t}\phi = \theta\sum_k \alpha_k e^{iE_k t}\psi_k = \sum_k \alpha_k e^{iE_k t}\theta\psi_k. \tag{A.6}$$

This equation together with (A.5) is incompatible with (A.2). If θ is assumed antiunitary, the same argument leads to $U_t\theta\phi = \theta U_{-t}\phi$, which precisely corresponds to (A.2) with the constant $\lambda_2 = 1$. Hence, equation (A.2) implies that the time inversion operator must be antiunitary. However, any antiunitary operator would satisfy this equation, and hence, (A.2) is not sufficient to determine the time inversion operator.

Notice that any antiunitary operator θ can be decomposed as

$$\theta = UK \tag{A.7}$$

where U is unitary and K is the complex conjugation. For any unitary operator U, it is convenient to introduce the notations

$$U^* \equiv KUK \quad \text{and} \quad U^T \equiv U^{\dagger *}, \tag{A.8}$$

which correspond to the usual complex conjugation and transpose when the Hilbert space is finite-dimensional.

The time inversion operator θ has an additional important property. From the physical requirement $\theta^2 = \lambda\mathbb{1}$, $\lambda \in \mathbb{C}$, and from the normal form (A.7), one gets

$$\lambda\mathbb{1} = \theta^2 = UKUK = UU^*. \tag{A.9}$$

Besides, $UU^\dagger = \mathbb{1}$ by unitarity. Hence, $U = \lambda U^{*-1} = \lambda U^T$, and, by taking the transpose $U^T = \lambda U$. We deduce that $\lambda = \pm 1$, that is

$$\theta^2 = \pm\mathbb{1}, \quad \text{or equivalently} \quad U^T = \pm U. \tag{A.10}$$

Making use of the identity (A.10), the equation (A.1) implies that a time-reversal invariant system satisfies

$$\theta U_t \theta = \pm U_{-t} \quad \text{for} \quad \theta^2 = \pm\mathbb{1}. \tag{A.11}$$

To summarize, if θ is an antiunitary operator commuting with the Hamiltonian and satisfying $\theta^2 = \pm\mathbb{1}$, then it satisfies (A.11). Conversely, if θ is an

antiunitary operator satisfying $\theta^2 = \pm\mathbb{1}$ and (A.11), then it commutes with the Hamiltonian. Indeed, if $\hat{h}$ stands for the Hamiltonian, we have

$$\begin{aligned} \theta U_t \theta = \pm U_{-t} \quad &\text{iff} \quad \pm\theta(1 - i\hat{h}t)\theta = 1 + i\hat{h}t \\ &\text{iff} \quad \pm\theta\hat{h}\theta = \hat{h} \\ &\text{iff} \quad [\hat{h}, \theta] = 0. \end{aligned} \tag{A.12}$$

Finally, notice that if θ is antiunitary, commutes with the Hamiltonian, and satisfies $\theta^2 = \pm\mathbb{1}$, so does $e^{i\varphi}\theta$. Hence, the time inversion operator is defined up to a phase.

A.1.2 Definition of the Time Inversion Operator

In this section, a time inversion operator is defined in the case of a system of particles without spin and in the case of a system of particles with spin 1/2. Defining a time inversion operator is equivalent to defining the unitary operator U appearing in the normal form (A.7). This has to be done in accordance with the algebra of observables considered and how time inversion is wanted to act on these observables.

In the context of time inversion, it is required to distinguish between three types of observables, that is between three types of self-adjoint operators.

- The operators $\hat{a}$ of type A, such that $\hat{a}\theta\phi = a\theta\phi$ as $\hat{a}\phi = a\phi$. Examples are the position $\hat{q}$ and the hamiltonian $\hat{h}$ of a time-reversal invariant system.

- The operators $\hat{b}$ of type B, such that $\hat{b}\theta\phi = -b\theta\phi$ as $\hat{b}\phi = b\phi$. Examples are the momentum $\hat{p}$, the angular momentum $\hat{l}_j$ and the spin $\hat{s}_j$.

- The operators $\hat{c}$ of type C that are non trivial sums of operators of type A and B. Their eigenvalues and eigenfunctions do not obey an easy transformation law. A physical example is the hamiltonian $\hat{h}$ of a particle in a magnetic field.

Lemma A.1 *A self-adjoint operator of type A commutes with θ, whereas a self-adjoint operator of type B anticommutes with θ.*

Proof. For sake of simplicity, we consider $\hat{a}$ with discrete spectrum only. Any state ϕ can then be decomposed $\phi = \sum_j \alpha_j \psi_j$, where $\{\psi_j\}$ is a complete orthonormal set of eigenfunctions of $\hat{a}$. We will write $\hat{a}\psi_j = a_j\psi_j$, with $a_j \in \mathbb{R}$. Since θ is antiunitary,

$$\theta\hat{a}\phi = \theta \sum_j \alpha_j a_j \psi_j = \sum_j \alpha_j^* a_j \theta\psi_j. \tag{A.13}$$

Besides, if $\hat{a}$ is of type A, then

$$\hat{a}\theta\phi = \hat{a} \sum_j \alpha_j^* \theta\psi_j = \sum_j \alpha_j^* a_j \theta\psi_j. \tag{A.14}$$

Therefore, $\theta\hat{a} = \hat{a}\theta$. Now, let $\hat{b}$ be a self-adjoint operator of type B and write $\hat{b}\psi_j = b_j\psi_j$ as before. By antiunitarity of θ, (A.13) with $\hat{b}$ in place of $\hat{a}$ and b_j in place of a_j still holds. But

$$\hat{b}\theta\phi = \hat{b} \sum_j \alpha_j^* \theta\psi_j = -\sum_j \alpha_j^* b_j \theta\psi_j. \tag{A.15}$$

This shows that $\theta\hat{b} = -\hat{b}\theta$, which finishes the proof.

□

A direct consequence of this lemma is that the product between two operators that are both of type A or both of type B yields an operator of type A, whereas the product between an operator of type A and one of type B is of type B.

Spinless Particles

Let us treat the case of spinless particles. For sake of simplicity, we consider a single particle in a one-dimensional space. The generalization to several particles in a higher dimensional euclidian space is then obvious. The algebra of observables for a particle without spin in one dimension is generated by the unity $\mathbb{1}$, the position $\hat{q}$, and the momentum $\hat{p}$. It is important to emphasize

that the subsequent calculations are performed in the position representation. It will be seen that another representation would lead to other possible choices for θ. Using the normal form $\theta = UK$, and the fact that $\hat{q}$ is of type A, we have that for all state ϕ

$$\begin{aligned} \theta\hat{q}\phi &= UKq\phi = Uq\phi^* \\ \theta\hat{q}\phi &= \hat{q}\theta\phi = qU\phi^* \quad \text{by lemma A.1} \end{aligned} \tag{A.16}$$

Besides, since $\hat{p}$ is of type B,

$$\begin{aligned} \theta\hat{p}\phi &= UK\frac{1}{i}\frac{\partial}{\partial x}\phi = -\frac{1}{i}U\frac{\partial}{\partial x}\phi^* \\ \theta\hat{p}\phi &= -\hat{p}\theta\phi = -\frac{1}{i}\frac{\partial}{\partial x}U\phi^* \quad \text{by lemma A.1} \end{aligned} \tag{A.17}$$

The relations (A.16) and (A.17) show that U commutes with $\hat{q}$ and $\hat{p}$ and therefore with any observable. This implies $U = c \cdot \mathbb{1}$, for some $c \in \mathbb{C}$ on the unit circle. Since θ is defined up to a phase, we can choose $U = \mathbb{1}$, and consequently

$$\theta = K. \tag{A.18}$$

It is important to notice that the choice $\theta = K$ is justified in the position representation but can be unappropriate in other representations. For example, in the momentum representation, the position and momentum operators read

$$(\hat{x}\psi)(p) = i\frac{\partial}{\partial p}\psi(p) \quad \text{and} \quad (\hat{p}\psi) = p\psi(p). \tag{A.19}$$

Calculations similar to (A.16) and (A.17) would show that the unitary operator of the normal form of θ must *anticommute* with $\hat{x}$ and $\hat{p}$.

Particles with Spin 1/2

Now, let us look at a system of particles with spin 1/2. In order to avoid cumbersome notations, the case of one particle of spin 1/2 in a one-dimensional euclidean space is first treated. The Hilbert space of such a system is the tensor product of the configuration space $L^2(\mathbb{R}, dx)$ and the spin space $\mathbb{C}^2$. It

is natural to define a time inversion operator that acts independently on each of these spaces,

$$\theta = \theta_c \otimes \theta_s, \tag{A.20}$$

where θ_c and θ_s are antiunitary operators on the configuration and the spin spaces respectively. The configuration time inversion operator is precisely the one that has been found in the previous section. In the position representation, we thus have $\theta_c = K$. The algebra of observables on the spin space is generated by the unity (type A) and the three Pauli matrices σ_1, σ_2 and σ_3 (type B). For $\alpha \in \{1,3\}$ and $z \in \mathbb{C}^2$,

$$\begin{aligned} \theta\sigma_\alpha z &= UK\sigma_\alpha z = U\sigma_\alpha z^* \\ \theta\sigma_\alpha z &= -\sigma_\alpha\theta z = -\sigma_\alpha U z^* \quad \text{by lemma A.1} \end{aligned} \tag{A.21}$$

The same calculation for σ_2 provides

$$\begin{aligned} \theta\sigma_2 z &= UK\sigma_2 z = U\sigma_2^* z^* = -U\sigma_2 z^* \\ \theta\sigma_2 z &= -\sigma_2\theta z = -\sigma_2 U z^* \quad \text{by lemma A.1} \end{aligned} \tag{A.22}$$

Therefore U must commute with σ_2 and anticommute with σ_1 and σ_3. This requirement is satisfied by $U = \sigma_2$, and consequently

$$\theta_s = \sigma_2 K. \tag{A.23}$$

For a system of particles with spin 1/2, the time inversion operator on the whole Hilbert space is merely

$$\theta = K \otimes \sigma_2 K \otimes \ldots \otimes K \otimes \sigma_2 K, \tag{A.24}$$

where the number of factors $K \otimes \sigma_2 K$ is equal to the number of particles. For a sytem of n particles with spin 1/2, θ satisfies

$$\theta^2 = (-1)^n \mathbb{1}. \tag{A.25}$$

A.1.3 Action on Observables and TR Invariance

In the previous sections, a time inversion operator θ has been built, depending on the algebra of observables considered. We now suppose that a Hilbert space and a time inversion operator $\theta = UK$ are given. One can associate with θ a corresponding action on the observables by imposing

$$(\theta\phi, A\theta\psi) = (\psi, A^\theta\phi). \tag{A.26}$$

After some algebra, one easily gets

$$A^\theta = U^T A^T U^*. \tag{A.27}$$

Two equivalent definitions of time-reversal invariance naturally follows from what has been exposed previously. The system is *TR invariant* if and only if one of the two conditions

- θ commutes with the Hamiltonian
- θ satisfies condition (A.11)

is fulfilled. In this case, (A.12) shows that the second condition is automatically satisfied. There is now a third equivalent definition of TR invariance, and this is precisely the one to be used for quantum graphs. The system is TR invariant if and only if

- $U_t^\theta = U_t$.

Indeed, for $\theta^2 = \pm\mathbb{1}$, or equivalently $U^T = \pm U$, we have

$$\begin{aligned} U_t^\theta = U_t \quad &\text{iff} \quad U^T U_t^T U^* = U_t \quad \text{iff} \quad U^\dagger U_t^\dagger U = U_t^* \\ &\text{iff} \quad U_{-t} = U U_t^* U^\dagger = U K U_t K U^\dagger \\ &\text{iff} \quad U_{-t} = U K U_t U^T K = \pm U K U_t U K = \pm\theta U_t \theta, \end{aligned} \tag{A.28}$$

which is precisely the condition (A.11).

The fact that the time inversion operator can either satisfy $\theta^2 = 1$ or $\theta^2 = -1$ naturally leads to the following definitions.

Definition A.2 *The set of quantum systems can be considered as the disjoint union of three different classes.*

- *The* class A *contains all quantum systems with broken TR invariance.*
- *The* class AI *contains all TR invariant systems with* $\theta^2 = 1$.
- *The* class AII *contains all TR invariant systems with* $\theta^2 = -1$.

The reader interested in a thinner classification of quantum systems (involving chirality) can refer to [44] and the literature cited in this source.

A.1.4 Magnetic Field and TR Invariance Breaking

In this paragraph, we show that adding a magnetic field to a TR invariant system breaks TR symmetry. Consider the Hamiltonian of a particle of mass $m = 1/2$ moving in a d-dimensional euclidean space subjected to a potential $V(\hat{q})$,

$$\hat{h} = \hat{p}^2 + V(\hat{q}). \tag{A.29}$$

A suitable time inversion operator is $\theta = K$, and since $\hat{q}$ and $\hat{p}^2$ are both of type A, lemma A.1 implies that $[\hat{h}, \theta] = 0$. Consequently, such a system is time-reversal invariant.

Now, suppose that a magnetic field is switched on. If $A(q)$ is a classical vector potential generating the magnetic field, the quantum Hamiltonian becomes

$$\hat{h}_A = (\hat{p} - eA(\hat{q}))^2 + V(\hat{q}). \tag{A.30}$$

The algebra of observables remains the same, and hence, $\theta = K$ is still valid. However, a direct calculation leads to

$$[\hat{h}_A, \theta] = 2e\theta\hat{p}A(\hat{q}), \tag{A.31}$$

showing that, unless $eA(q) = 0$, the time-reversal symmetry of this system is broken.

A.2 Time Inversion on Quantum Graphs

In this section, the action of time inversion on quantum graphs is explicitely deduced from the previous general considerations. Then, we expose how time-reversal invariance can be dealt with in the field theoretical description of quantum graphs.

A.2.1 Time Inversion in the Amplitude Space

The Hamiltonian considered on the graph is the one of a free particle on each bond. We restrict our attention to the case of a spinless particle, and therefore, the corresponding time inversion operator over $\mathbb{R}$ would merely be $\theta = K$. Time-reversal invariance could directly be broken by the introduction of some magnetic field on the graph. However, in the absence of magnetic field, it can still be broken by the boundary conditions specified at the vertices.

Firstly, it is natural to impose on the time inversion operator θ to act separately on each bond, that is

$$\bigoplus_{b=1}^{B} \psi_b \quad \mapsto \quad \bigoplus_{b=1}^{B} \theta\psi_b. \tag{A.32}$$

Then, on each bond b, we have $\theta = K$, so that

$$\begin{aligned} \theta\psi_b &= \theta\left(a_{b+}e^{ik(x-\frac{L_b}{2})} + a_{b-}e^{-ik(x-\frac{L_b}{2})}\right) \\ &= a_{b+}^{*}e^{-ik(x-\frac{L_b}{2})} + a_{b-}^{*}e^{ik(x-\frac{L_b}{2})} \end{aligned} \tag{A.33}$$

One can deduce form this map an action $\mathcal{T}$ of time inversion on the amplitudes space $\mathcal{A}$. It reads

$$|a\rangle \quad \mapsto \quad \mathcal{T}|a\rangle = \sigma_1^d K|a\rangle, \tag{A.34}$$

where σ_1^d is the first Pauli matrix acting in the direction space. Notice that the action $\mathcal{T}$ obtained is an antiunitary operator on the amplitudes space $\mathcal{A}$.

This action $\mathcal{T}$ on physical states defines an action on the algebra of observables by the identity

$$\langle \mathcal{T}e_\beta|A|\mathcal{T}e_{\beta'}\rangle = \langle e_{\beta'}|A^{\mathcal{T}}|e_{\beta'}\rangle, \tag{A.35}$$

for all $\beta \in \mathbb{N}_{2B}$ and for all matrix $A \in L(\mathcal{A})$. One easily get

$$A \quad \mapsto \quad A^{\mathcal{T}} \equiv \sigma_1^d A^T \sigma_1^d. \tag{A.36}$$

Time-reversal invariance for graphs is defined similarly as in A.1.3 for a system of particles in $\mathbb{R}^d$. A graph is said to be time-reversal invariant if its evolution matrix $U(k)$ is left invariant under time inversion. This means that for all $k \geq 0$,

$$\sigma_1^d T(k) S^T T(k) \sigma_1^d = T(k) S T(k). \tag{A.37}$$

Setting $k = 0$ first, and then letting this parameter vary shows that this identity can only be realized if

$$\sigma_1^d T(k) \sigma_1^d = T(k) \quad \text{and} \quad \sigma_1^d S^T \sigma_1^d = S. \tag{A.38}$$

The first condition fails if and only if a magnetic field is considered. Indeed, a magnetic field makes $T(k)$ non trivial in the direction space $\mathcal{A}_d$. The second equation provides a condition on the scattering matrix S, that is on the boundary conditions at the vertices. It can also be written

$$\langle e_{b',d'}|S|e_{b,d}\rangle = \langle e_{b,-d}|S^*|e_{b',-d'}\rangle. \tag{A.39}$$

Neumann boundary conditions are thus an example of TR invariant boundary conditions.

A.2.2 The Time-Reversal Space

A convenient way to deal with TR invariance in the field theoretical approach to quantum graphs is to introduce a time-reversal space. This space, denoted by TR, is the 2-dimensional $\mathbb{C}$-linear space spanned by $\{\uparrow, \downarrow\}$. Let us consider the space $X = \mathcal{A} \otimes \mathbb{C}^n$, where $\mathcal{A}$ is the amplitude space. In practice, the two interesting cases are $n = 1$, in which case $X = \mathcal{A}$, and $n = 2$ with $\mathbb{C}^2$ standing for the retarded-advanced space RA. We introduce a Grassmann algebra Λ and the Grassmann envelope $(X \oplus X)(\Lambda)$ as defined in Appendix B. With any supervector $\psi \in (X \oplus X)(\Lambda)$, we associate a supervector

$\Psi \in (X \otimes TR \oplus X \otimes TR)(\Lambda)$ defined by

$$\psi \mapsto \Psi \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \psi \\ T\psi \end{pmatrix}_{TR} = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi \\ \sigma_1^d \psi^* \end{pmatrix}_{TR} . \tag{A.40}$$

The new supervector Ψ is referred to as the *time-reversal doubling of* ψ, and the index TR indicates that the matrix structure is written in the time-reversal space. Notice that in (A.40), time inversion merely stands for complex conjugation on the component $\mathbb{C}^n$ of X, while it acts on $\mathcal{A}$ according to (A.34). That is the reason why the symbol T is kept.

Similarly, one can also associate with each matrix $N \in L(X|X)$ a new matrix $\mathcal{N} \in L(X \otimes TR|X \otimes TR)$ defined by

$$N \mapsto \mathcal{N} \equiv \begin{pmatrix} N & 0 \\ 0 & N^T \end{pmatrix}_{TR} = \begin{pmatrix} N & 0 \\ 0 & \sigma_1^d N^T \sigma_1^d \end{pmatrix}_{TR} , \tag{A.41}$$

and the supermatrix $\mathcal{N}$ is referred to as the *time-reversal doubling of* N. In fact, the Bose-Bose blocks of N and N^T have the same characteristic polynomial, and consequently the same eigenvalues. Indeed,

$$\begin{aligned} \det\left(t\mathbf{1} - \left(\sigma_1^d \Lambda^T \sigma_1^d\right)_{BB}\right) &= \det\left(t\mathbf{1} - \sigma_1^d {\Lambda_{BB}}^T \sigma_1^d\right) \\ &= \det\left(\left(\sigma_1^d t\mathbf{1} \sigma_1^d\right)^T - \Lambda_{BB}\right) \\ &= \det\left(t\mathbf{1} - \Lambda_{BB}\right) \end{aligned} \tag{A.42}$$

Since we work with the convention $\chi^{**} = -\chi$ for any odd element of Λ, the complex conjugate and the hermitian conjugate of Ψ in (A.40) have to be deduced with care. They read

$$\Psi^* = \frac{1}{\sqrt{2}} \begin{pmatrix} \Psi^* \\ \sigma_1^d \sigma_3^{BF} \Psi \end{pmatrix}_{TR} \quad \text{and} \quad \Psi^\dagger = \frac{1}{\sqrt{2}} \left(\Psi^\dagger, \Psi^T \sigma_3^{BF} \sigma_1^d\right)_{TR} . \tag{A.43}$$

The presence of σ_3^{BF} leads to some entanglements between the time-reversal and the Bose-Fermi spaces.

Lemma A.3 *Let ψ_1, ψ_2 be supervectors in $(X \oplus X)(\Lambda)$, and let $N \in L(X|X)$. Denote by Ψ_1, Ψ_2 and $\mathcal{N}$ their time-reversal doubles. Then,*

$$\Psi_1^\dagger \mathcal{N} \Psi_2 = \frac{1}{2}(\psi_1^\dagger N \psi_2 + \psi_2^\dagger N \psi_1).$$

Proof. The second formula in (A.43) implies that

$$\Psi_1^\dagger \mathcal{N} \Psi_2 = \frac{1}{2}\left(\psi_1^\dagger N \psi_2 + \psi_1^T \sigma_3^{BF} N^T \psi_2^*\right), \tag{A.44}$$

and it therefore suffices to prove that the second term here is equal to the second term in the right-hand side of the lemma. Let i, j be indices for the components in the space X and $s, s' \in \{B, F\}$. Then,

$$\begin{aligned}
\psi_1^T \sigma_3^{BF} N^T \psi_2^* &= \sum_{i,j,s,s'} (-1)^s \psi_{1is} (N^T)_{is,js'} \psi^*_{2js'} \\
&= \sum_{i,j,s,s'} (-1)^s \psi_{1is} (-1)^{s'(s+1)} N_{js',is} \psi^*_{2js'} \\
&= \sum_{i,j,s,s'} (-1)^{s+s'(s+1)} (-1)^{s(s+s')} N_{js',is} \psi_{1is} \psi^*_{2js'} \\
&= \sum_{i,j,s,s'} (-1)^{s'} A_{js',is} (-1)^{ss'} \psi^*_{2js'} \psi_{1is} \\
&= \sum_{i,j,s,s'} (-1)^{s'+ss'} (-1)^{s'(s+s')} \psi^*_{2js'} N_{js',is} \psi_{1is} \\
&= \sum_{i,j,s,s'} \psi^*_{2js'} N_{js',is} \psi_{1is} \;=\; \psi_2^\dagger N \psi_1
\end{aligned} \tag{A.45}$$

□

Corollary A.4 *The most important case in the previous lemma is $\psi_1 = \psi_2$, which yields*

$$\Psi^\dagger \mathcal{N} \Psi = \psi^\dagger N \psi.$$

Lemma A.5 *Let $N \in L(X|X)$ be a supermatrix defining a convergent Gaussian superintegral*

$$I \equiv \int D\psi \; exp\left(-\psi^\dagger N \psi\right).$$

Then, after TR doubling, the Gaussian superintegral becomes

$$\int D\Psi \; exp\left(-\Psi^\dagger \mathcal{N} \Psi\right) = \left(\int D\psi \; exp\left(-\Psi^\dagger \mathcal{N} \Psi\right)\right)^2 = I^2.$$

Proof. The second equality of the lemma directly follows from Corollary A.4. For the first equality, one first notices that $\mathcal{N}$ is diagonal in TR. Moreover, (A.42) ensures that the eigenvalues of $\mathcal{N}_{BB}$ are precisely the eigenvalues of

N_{BB}, but the first are twice more degenerated. Therefore, $\mathcal{N}$ also defines a convergent Gaussian superintegral, and

$$\int D\Psi \exp\left(-\Psi^\dagger \mathcal{N} \Psi\right) = \mathrm{sdet}\mathcal{N}^{-1} = \mathrm{sdet}N^{-2} = I^2 \tag{A.46}$$

□

The time-reversal doublings introduced in (A.40) and (A.41) call for a new definition of duality for supermatrices in $L(X \otimes TR | X \otimes TR)$. Let us introduce the supermatrix

$$\tau \equiv \sigma_1^d \begin{pmatrix} \sigma_1^{TR} & 0 \\ 0 & -i\sigma_2^{TR} \end{pmatrix}_{BF} = \sigma_1^d \begin{pmatrix} 0 & \sigma_3^{BF} \\ 1\!\!1_{BF} & 0 \end{pmatrix}_{TR}. \tag{A.47}$$

This supermatrix is invertible, and its inverse reads $\tau^{-1} = \sigma_3^{BF}\tau$, that is

$$\tau^{-1} \equiv \sigma_1^d \begin{pmatrix} \sigma_1^{TR} & 0 \\ 0 & i\sigma_2^{TR} \end{pmatrix}_{BF} = \sigma_1^d \begin{pmatrix} 0 & 1\!\!1_{BF} \\ \sigma_3^{BF} & 0 \end{pmatrix}_{TR}. \tag{A.48}$$

It can be readily checked that $\tau^T = \tau^{-1}$.

Lemma A.6 *Let $\Psi \in (X \otimes TR \oplus X \otimes TR)(\Lambda)$ be a supervector obtained from some $\psi \in (X \oplus X)(\Lambda)$ by time-reversal doubling. Then*

$$\Psi = \tau\Psi^* \quad \textit{and} \quad \Psi^\dagger = \Psi^T\tau.$$

Proof. First, notice that the second relation is obtained from the first by transposition. In order to get the first equality, we compare the components of Ψ^* given in (A.43) with the components of Ψ. This gives

$$\Psi_\uparrow = \frac{1}{\sqrt{2}}\psi = \sigma_1^d\sigma_3^{BF}\Psi_\downarrow^* \tag{A.49}$$

$$\Psi_\downarrow = \frac{1}{\sqrt{2}}\sigma_1^d\psi^* = \sigma_1^d\Psi_\uparrow^*. \tag{A.50}$$

These equations, once written in a matrix form, yield the first equality of the lemma

□

Let $\mathcal{N}$ be any supermatrix in $L(X\otimes TR|X\otimes TR)$. The *generalized transpose* $\mathcal{N}^\tau$ of $\mathcal{N}$ is defined by

$$\mathcal{N}^\tau \equiv \tau \mathcal{N}^T \tau^{-1}, \tag{A.51}$$

where $\mathcal{N}^T$ denotes the usual transpose of $\mathcal{N}$. This operation is tailored in order to get the following property.

Lemma A.7 *Let $\mathcal{N}$ be any supermatrix in $L(X \otimes TR|X \otimes TR)$, and let $\Psi_1, \Psi_2 \in (X \otimes TR \oplus X \otimes TR)(\Lambda)$ be supervectors obtained by time-reversal doubling. Then,*

$$\Psi_1^\dagger \mathcal{N} \Psi_2 = \Psi_2^\dagger \mathcal{N}^\tau \Psi_1.$$

Proof. Let i, j be indices for the components in the space X and s, s' be indices in Bose-Fermi space. A direct calculation yields

$$\begin{aligned}
\Psi_1^\dagger \mathcal{N} \Psi_2 &= \sum_{i,j,s,s'} \Psi^*_{1is} \mathcal{N}_{is,js'} \Psi_{2js'} \\
&= \sum_{i,j,s,s'} (-1)^{s(s'+1)} \Psi^*_{1is} (\mathcal{N}^T)_{js',is} \Psi_{2js'} \\
&= \sum_{i,j,s,s'} (-1)^{s'} \Psi_{2js'} (\mathcal{N}^T)_{js',is} \Psi^*_{1is} \\
&= \Psi_2^T \sigma_3^{BF} \mathcal{N}^T \Psi_1^*.
\end{aligned} \tag{A.52}$$

But, by Lemma A.6, $\Psi_2^T = \Psi_2^\dagger \tau^{-1}$ and $\Psi_1^* = \tau^{-1}\Psi_1$. Injecting these identities into the previous equation, and making use of $\tau^{-1}\sigma_3^{BF} = \tau$, gives

$$\Psi_1^\dagger \mathcal{N} \Psi_2 = \Psi_2^\dagger \tau^{-1} \sigma_3^{BF} \mathcal{N}^T \tau^{-1} \Psi_1 = \Psi_2^\dagger \tau \mathcal{N}^T \tau^{-1} \Psi_1 = \Psi_2^\dagger \mathcal{N}^\tau \Psi_1 \tag{A.53}$$

□

Remark A.8 *The following properties of the generalized transposition are direct consequences of the previous results.*

1. $(\mathcal{N}_1\mathcal{N}_2)^\tau = \mathcal{N}_2^\tau \mathcal{N}_1^\tau$ *and* $\mathcal{N}^{\tau\tau} = \mathcal{N}$
2. $(\mathcal{N}^\tau)^* = (\mathcal{N}^*)^\tau$, $(\mathcal{N}^\tau)^T = (\mathcal{N}^T)^\tau$, *and* $(\mathcal{N}^\tau)^\dagger = (\mathcal{N}^*)^\dagger$
3. *If $\mathcal{N}$ is obtained by time-reversal doubling, then, $\mathcal{N}^\tau = \mathcal{N}$.*

Appendix B

The Bose-Fermi Space

The goal of this appendix is twofold. Firstly, it can be considered as an extremely short introduction to Grassmann algebras, $\mathbb{Z}_2$-graded linear spaces over these algebras, and as a presentation of some useful analytical results. However, it can by no means be substituted for a more complete treatment, like the one in the excellent textbook [8] for example. The main goal of this appendix is rather to clearly define the conventions used throughout this work. These conventions, although frequently used, are not completely standard, and there is rather a multitude of equivalent but different conventions that can be found in the literature.

B.1 Supervectors and Supermatrices

Let us introduce a set $\chi = \{\chi_k : k \in \mathbb{N}_D\}$, with $D \geq 1$. The *complex D-dimensional Grassmann algebra* Λ is the set of all the polynomials in $\mathbb{C}[\chi]$ with the additional rule

$$\chi_k\chi_l + \chi_l\chi_k = 0 \in \mathbb{C}. \tag{B.1}$$

The elements in χ are called *generators* of Λ. In fact, it can be seen that Λ has many different families of generators. It has thus an intrinsic existence, independent of χ. The anticommutation relations (B.1) imply in particular that any generator χ_k satisfies $\chi_k^2 = 0$. Hence, any polynomial $f \in \Lambda$ can be

expressed as

$$f(\chi) = \sum_{p=0}^{D} \sum_{k_1<\ldots<k_p} f_{k_1,\ldots,k_p} \chi_{k_1} \cdots \chi_{k_p}, \tag{B.2}$$

for some suitable coefficients $f_{k_1,\ldots,k_p} \in \mathbb{C}$. Besides, a Grassmann algebra Λ can be decomposed as a direct sum

$$\Lambda = \Lambda_{\bar{0}} \oplus \Lambda_{\bar{1}}, \tag{B.3}$$

where $\Lambda_{\bar{0}}$ is the subset of elements commuting with any other elements in Λ. The decomposition (B.3) makes Λ a $\mathbb{Z}_2$-graded linear space. If $f(\chi) \in \Lambda_{\bar{0}}$, all its terms are products of even numbers of generators. Hence, $\Lambda_{\bar{1}}$ can be chosen as the set of polynomials whose terms only contain odd numbers of generators. In fact, the commuting elements $\Lambda_{\bar{0}}$ do not depend on the particular choice of generating family χ, whereas $\Lambda_{\bar{1}}$ as defined above does. One also has the rule $\Lambda_{\bar{i}}\Lambda_{\bar{j}} \subset \Lambda_{\overline{i+j}}$ for $i, j \in \{0, 1\}$, where the addition is performed in $\mathbb{Z}_2$. This property makes Λ a $\mathbb{Z}_2$-graded algebra. Any Grassmann algebra Λ possesses an homomorphism m, which associates with each element $f(\chi)$ the complex number $f(0)$ obtained by sending all the Grassmann generators to zero. This element $(mf)(\chi) = f(0)$ is called the *body* of the Grassmann number $f(\chi)$, whereas the remainder $f(\chi) - f(0)$ is referred to as the *soul*. These two notions are independent of the generating family χ.

Let X be a n-dimensional $\mathbb{C}$-linear space with basis $\{e_i\}$. The set $X \oplus X$ has elements of the form

$$V = \sum_{i=1}^{n} V_0^i e_i \oplus \sum_{i=1}^{n} V_1^i e_i' \tag{B.4}$$

where $\{e_i'\}$ is the same basis as $\{e_i\}$, and the coefficients $V_0^i, V_1^i \in \mathbb{C}$. This is also a $\mathbb{Z}_2$-graded linear space. One can now consider the expressions of the type (B.4) where the coefficients V_0^i belong to $\Lambda_{\bar{0}}$ and the coefficients V_1^i belong to $\Lambda_{\bar{1}}$. The resulting objects are of the form

$$V(\chi) = \sum_{i=1}^{n} V_B^i(\chi) e_i \oplus \sum_{i=1}^{n} V_F^i(\chi) e_i', \quad \text{where} \quad V_B(\chi) \in \Lambda_{\bar{0}},\ V_F(\chi) \in \Lambda_{\bar{1}}, \tag{B.5}$$

and are called *supervectors*. The set of such supervectors, denoted by $(X \oplus X)(\Lambda)$ is sometimes referred to as the *Grassmann envelope* of $X \oplus X$. Once the basis $\{e_i\}$ is fixed, $V(\chi)$ in (B.5) is represented by

$$V = \begin{pmatrix} V_B \\ V_F \end{pmatrix}, \quad \text{with} \quad V_B = \begin{pmatrix} V_B^1 \\ \vdots \\ V_B^n \end{pmatrix}, \quad V_F = \begin{pmatrix} V_F^1 \\ \vdots \\ V_F^n \end{pmatrix}. \tag{B.6}$$

In the last formulae and in the sequel, the dependency on the generators χ are kept implicit. The 2-dimensional structure written in (B.6) is called *Bose-Fermi* space, and the first and second sectors are respectively called *bosonic* and *fermionic*. In the literature, the set of supervectors (B.6) is sometimes denoted by $\Lambda^{n|n}$. However, the notation $(X \oplus X)(\Lambda)$ is often preferred since it explicitely refers to the underlying space X and not only to its dimension. The body of a supervector mV is the complex vector obtained by only retaining the bodies of all its components.

We endow the Grassmann algebra Λ with an involution $f \mapsto f^*$, characterized by the rules

$$\begin{aligned} &(f_1 + f_2)^* = f_1^* + f_2^* && (f_1 \cdot f_2)^* = f_1^* \cdot f_2^*, \\ &(a + ib)^* = a - ib \text{ for } a, b \in \mathbb{R} && \chi_k^{**} = -\chi \text{ if } \chi_k \in \chi. \end{aligned} \tag{B.7}$$

The element f^* is naturally called *complex conjugate* of f. The rules (B.7) show that f^{**} is equal to f if $f \in \Lambda_0$, and to $-f$ if $f \in \Lambda_1$. Moreover, the choice $\chi_k^{**} = -\chi_k$ implies $(\chi_k^* \chi_k)^* = \chi_k^* \chi_k$. Although such an involution is commonly used in the literature, its introduction is not necessary, and all the calculations can be performed without referring to this notion. However, it enables to generalize the concepts of unitarity and hermiticity.

For a supervector V, V^* denotes the supervector containing the complex conjugates of the components of V. Beside the involution, two other operations on supervectors are needed. Let $V \in (X \oplus X)(\Lambda)$ be as in (B.6). Its transpose and its hermitian conjugate are respectively defined by

$$\begin{aligned} V^T &\equiv (V_B{}^T \; V_F{}^T) \quad , \quad V_{B,F}{}^T \equiv (V_{B,F}^1 \; \ldots V_{B,F}^n) \\ V^\dagger &\equiv (V_B{}^\dagger \; V_F{}^\dagger) \quad , \quad V_{B,F}{}^\dagger \equiv (V_{B,F}^{1*} \; \ldots V_{B,F}^{n*}) \end{aligned} \tag{B.8}$$

Hence, one can write $V^\dagger = V^{*T} = V^{T*}$. However, because of the convention (B.7) for the involution, we have the unusual rule $V^{\dagger\dagger} = \sigma_3^{BF} V$, where σ_3^{BF} is the third Pauli matrix acting on the Bose-Fermi space.

In order to preserve the commuting and antcommuting properties of the Bose and Fermi sectors, an endomorphism A on the set of supervectors $(X \oplus X)(\Lambda)$ must be of the form

$$A = \left(\begin{array}{c|c} A_{BB} & A_{BF} \\ \hline A_{FB} & A_{FF} \end{array}\right) \tag{B.9}$$

where each block is $n \times n$, the coefficients of A_{BB} and A_{FF} are in $\Lambda_{\bar{0}}$, and the coefficients of A_{BF} and A_{FB} are in $\Lambda_{\bar{1}}$. In the writing (B.9), a particular basis for the underlying space X has been assumed. Such endomorphisms are then called *supermatrices*, and the set of all supermatrices is written $L(X|X)$. In practice, each underlying space considered has a privileged basis, and therefore the writing (B.9) is unambiguous. Notice that an endomorphism A can be written as the direct sum of its diagonal and off-diagonal blocks. This shows that $L(X|X)$ is also a $\mathbb{Z}_2$-graded algebra. The body mA of A is the supermatrix obtained from replacing all the components with their bodies.

We define the *superdeterminant* of $A \in L(X|X)$ by

$$\mathrm{sdet}A \equiv \det\left[A_{BB} - A_{BF}{A_{FF}}^{-1}A_{FB}\right] \cdot \det\left[A_{FF}\right]^{-1} \tag{B.10}$$

The matrices in the two square brackets have commuting components, and their determinants have to be understood as usual determinants. Notice that this definition is meaningful when $\det A_{FF} \neq 0$. It is not difficult to check that $\mathrm{sdet}AB = \mathrm{sdet}A \cdot \mathrm{sdet}B$. The *supertrace* of A is

$$\mathrm{str}A \equiv \mathrm{tr}A_{BB} - \mathrm{tr}A_{FF}. \tag{B.11}$$

It is related to the superdeterminant through the fundamental formula

$$\mathrm{sdet}A = e^{\mathrm{str}\log A}. \tag{B.12}$$

The adjoint $A^\dagger$ and the transpose A^T of a supermatrix $A \in L(X|X)$ are defined by the equations

$$(AV_1)^\dagger V_2 = V_1^\dagger A^\dagger V_2 \quad \text{and} \quad (AV_1)^T V_2 = V_1^T A^T V_2 \tag{B.13}$$

for all $V_1, V_2 \in (X \oplus X)(\Lambda)$. A short calculation shows that, if A is given by (B.9), then

$$A^\dagger = \left(\begin{array}{c|c} A_{BB}{}^\dagger & A_{FB}{}^\dagger \\ \hline -A_{BF}{}^\dagger & A_{FF}{}^\dagger \end{array}\right) \quad \text{and} \quad A^T = \left(\begin{array}{c|c} A_{BB}{}^T & A_{FB}{}^T \\ \hline -A_{BF}{}^T & A_{FF}{}^T \end{array}\right) \tag{B.14}$$

where in each block † and ^{T} denote the usual hermitian conjugate and transpose. In particular, (B.14), together with the definition (B.7) of the involution, shows that $A^{\dagger\dagger} = A$, but $A^{TT} = \sigma_3^{BF} A \sigma_3^{BF}$.

B.2 Grassmann Analysis

B.2.1 Grassmann Analytic Functions

In the previous section, the Grassmann algebra Λ has been defined, and some linear algebra over Λ has been introduced. In Grassmann analysis, the main ingredients are the so-called Grassmann analytic functions. In order to define these functions, the set of variables on which they act has first to be described. Let U be a domain in $\mathbb{C}^d$. The *variable algebra* $\Lambda(U)$ consists of the elements

$$v(x, \chi) = \sum_{p=0}^{D} \sum_{k_1 < \ldots < k_p} v_{k_1,\ldots,k_p}(x)\, \chi_{k_1} \cdots \chi_{k_p}, \tag{B.15}$$

where the complex functions $v_{k_1,\ldots,k_p}$ are analytic in U. This algebra contains two types of distinguished variables. The *even*, or *commuting*, *variables* are of the form

$$f(x, \chi) = \sum_{p \text{ even}} \sum_{k_1 < \ldots < k_p} f_{k_1,\ldots,k_p}(x)\, \chi_{k_1} \cdots \chi_{k_p} \tag{B.16}$$

and they compose the set $\Lambda^{1|0}(U)$. The *odd*, or *anticommuting*, *variables* assume the form

$$\varphi(x, \chi) = \sum_{p \text{ odd}} \sum_{k_1 < \ldots < k_p} \varphi_{k_1,\ldots,k_p}(x)\, \chi_{k_1} \cdots \chi_{k_p} \tag{B.17}$$

and they form the set $\Lambda^{0|1}(U)$. When x runs over U, the variables $f(x,\chi)$ and $\varphi(x,\chi)$ indeed vary in $\Lambda^{1|0}(U)$ and $\Lambda^{0|1}(U)$. Moreover, the identification of $\Lambda(U)$ with $\Lambda^{1|0}(U) \oplus \Lambda^{0|1}(U)$ makes $\Lambda(U)$ a $\mathbb{Z}_2$-graded algebra. However, it is often required to work with several variables, and this necessitates the introduction of different variable sets,

$$\Lambda^{n|m}(U) \equiv \left(\Lambda^{1|0}(U)\right)^n \oplus \left(\Lambda^{0|1}(U)\right)^m. \tag{B.18}$$

Basically, an element of $\Lambda^{n|m}(U)$ assigns a supervector to any point in U. Besides, notice that a n-tuple $\boldsymbol{f} = (f_1, \ldots, f_n) \in \Lambda^{n|0}(U)$ of commuting variables can always be split as $\boldsymbol{f} = m\boldsymbol{f} + \boldsymbol{s}$, where $m\boldsymbol{f} \equiv (mf_1, \ldots, mf_n)$ contains the bodies of the variables, and the remainder $\boldsymbol{s}$ contains the souls. Finally, the *spectrum* of $\boldsymbol{f}$, denoted by $\mathrm{Spec}(\boldsymbol{f})$, is defined as being the range of $m\boldsymbol{f}$.

The Grassmann analytic functions are objects of the form $F(\boldsymbol{f}, \boldsymbol{\varphi})$, where $\boldsymbol{f} \in \Lambda^{n|0}(U)$ and $\boldsymbol{\varphi} \in \Lambda^{0|m}(U)$. They are defined in two steps. Let $V \subset \mathbb{C}^n$ be an open domain, and let G be a complex analytic function on V. One can define

$$G(\boldsymbol{f}) \equiv \sum_{q_1,\ldots,q_n \geq 0} \frac{\partial_1^{q_1} \ldots \partial_n^{q_n}}{q_1! \ldots q_n!} G(m\boldsymbol{f}) s_1^{q_1} \ldots s_n^{q_n}, \tag{B.19}$$

for all $\boldsymbol{f} = m\boldsymbol{f} + \boldsymbol{s}$ in $\Lambda^{n|0}(U)$ such that $\mathrm{Spec}(\boldsymbol{f}) \subset V$. Since the souls in $\boldsymbol{s}$ are nilpotents, the series stops. Now, for $\boldsymbol{f} \in \Lambda^{n|0}(U)$ and $\boldsymbol{\varphi} \in \Lambda^{0|m}(U)$, the expression

$$F(\boldsymbol{f}, \boldsymbol{\varphi}) = \sum_{q=0}^{m} \sum_{i_1 < \ldots < i_q} F_{i_1,\ldots i_q}(\boldsymbol{f}) \varphi_{i_1} \ldots \varphi_{i_q}, \tag{B.20}$$

is called *Grassmann analytic function* if each $F_{i_1,\ldots i_q}$ is as in (B.19). A Grassmann analytic function thus assigns an element of $\Lambda(U)$ to each element of the variable set $\Lambda^{n|m}(U)$. Moreover, it is said to be *even* (resp. *odd*) if the polynomial (B.20) only involves products of even (resp. odd) numbers of anticommuting variables φ_j. Besides, it is said to have *compact support* if all the complex functions $F_{i_1,\ldots i_q} \circ m$ vanish on the boundary of $\mathrm{Spec}(\boldsymbol{f})$.

Let $\boldsymbol{f} \in \Lambda^{n|0}(U)$ and $\boldsymbol{\varphi} \in \Lambda^{0|m}(U)$. Their union $\boldsymbol{f} \cup \boldsymbol{\varphi} \in \Lambda^{n|m}(U)$ is said to *generate* a subalgebra $L \subset \Lambda(U)$ if and only if any element $v \in L$ can be seen

as a Grassmann analytic function of (f, φ) as in (B.20).

There is also a concept of derivative in Grassmann analysis. If $F(f, \varphi)$ is a Grassmann analytic function, one wants to write a small variation of F as

$$dF = \frac{\partial}{\partial f_j} F(f,\varphi) \cdot df_j + F(f,\varphi) \frac{\overleftarrow{\partial}}{\partial \varphi_j} \cdot d\varphi_j \tag{B.21}$$

$$= df_j \cdot \frac{\partial}{\partial f_j} F(f,\varphi) + d\varphi_j \cdot \frac{\overrightarrow{\partial}}{\partial \varphi_j} F(f,\varphi) \tag{B.22}$$

where the summation convention is implicit. The derivative operator d is also required to be linear and to satisfy some generalized Leibnitz rule. The derivative $\partial_{f_j} F$ with respect to the commuting variable f_i is simply defined by replacing the functions $F_{i_1,\ldots i_q}$ in (B.20) with the corresponding derivatives $\partial_j F_{i_1,\ldots i_q}$, which in turns come from considering the complex function $\partial_j F_{i_1,\ldots i_q} \circ m$ in (B.19) in place of $F_{i_1,\ldots i_q} \circ m$. The *right* and *left derivatives* of $F(f, \varphi)$ with respect to an anticommuting variable φ_j are respectively defined by

$$\varphi_{i_1} \cdots \varphi_{i_p} \frac{\overleftarrow{\partial}}{\partial \varphi_j} = \sum_{k=1}^{p} \delta_{j,i_k} (-1)^{p-k} \varphi_{i_1} \cdots \hat{\varphi}_{i_k} \cdots \varphi_{i_p} \tag{B.23}$$

$$\frac{\overrightarrow{\partial}}{\partial \varphi_j} \varphi_{i_1} \cdots \varphi_{i_p} = \sum_{k=1}^{p} \delta_{j,i_k} (-1)^{k-1} \varphi_{i_1} \cdots \hat{\varphi}_{i_k} \cdots \varphi_{i_p} \tag{B.24}$$

and by linearity on the expression (B.20) for F. In other words, in (B.23) φ_j is brought to the right of the monomial and then erased, whereas in (B.24) φ_j is brought to the left of the monomial. It is easy to see that right and left derivatives are the same if F is odd, and are opposite to each other if F is even.

B.2.2 The Berezin Integral

The goal is to integrate Grassmann analytic functions $F : \Lambda^{n|m}(U) \to \Lambda(U)$, where U is a domain in $\mathbb{C}^d$. First, let $G : \Lambda^{n|0}(U) \to \Lambda(U)$ be Grassmann analytic. Its superintegral is defined by

$$\int G(f) Df \equiv \int_{\mathrm{Spec}(f)} G(x) \prod_{i=1}^{n} \frac{dx_i^* dx_i}{\pi}, \tag{B.25}$$

where the right-hand side has to be understood as an usual complex integral. Notice that this definition basically treats the commuting variables f_i as usual complex variables. Let us now turn to the superintegral of a more general Grassmann analytic function $F : \Lambda^{n|m}(U) \to \Lambda(U)$. If F is given by (B.20), one defines

$$\int F(\boldsymbol{f},\boldsymbol{\varphi})D\boldsymbol{f}D\boldsymbol{\varphi} \equiv \int F_{1,\dots,m}(\boldsymbol{f})D\boldsymbol{f}, \tag{B.26}$$

and the right-hand side is defined as in (B.25). In fact, the definition (B.26) amounts to work with the measure $D\boldsymbol{\varphi} = d\varphi_1 \dots d\varphi_m$ and the *Berezin's rules*

$$\int \varphi_k d\varphi_k = 1, \quad \int d\varphi_k = 0. \tag{B.27}$$

Notice that integrating an expression with respect to an anticommuting variable φ_k amounts to differentiate this expression with respect to this variable.

Let $(\boldsymbol{f},\boldsymbol{\varphi}),(\boldsymbol{g},\boldsymbol{\psi}) \in \Lambda^{n|m}(U)$ be two systems of generators of a subalgebra $L \subset \Lambda(U)$, and define

$$\Delta\left(\frac{\boldsymbol{f},\boldsymbol{\varphi}}{\boldsymbol{g},\boldsymbol{\psi}}\right) \equiv \mathrm{sdet}\begin{pmatrix} \left(\frac{\partial}{\partial g_j} f_i\right)_{(i,j)\in\mathbb{N}_n\times\mathbb{N}_n} & \left(f_i \frac{\overleftarrow{\partial}}{\partial \psi_j}\right)_{(i,j)\in\mathbb{N}_n\times\mathbb{N}_m} \\ \left(\frac{\partial}{\partial g_j}\varphi_i\right)_{(i,j)\in\mathbb{N}_m\times\mathbb{N}_n} & \left(\frac{\partial}{\partial \psi_j}\varphi_i\right)_{(i,j)\in\mathbb{N}_m\times\mathbb{N}_m} \end{pmatrix}. \tag{B.28}$$

This quantity is called *Berezinian* of the change of variables $(\boldsymbol{f},\boldsymbol{\varphi}) \to (\boldsymbol{g},\boldsymbol{\psi})$, and it plays the role of Jacobian in superintegrals. Indeed, if $F(\boldsymbol{f},\boldsymbol{\varphi})$ is Grassmann analytic with compact support, then

$$\int F(\boldsymbol{f},\boldsymbol{\varphi})D\boldsymbol{f}D\boldsymbol{\varphi} = \int F\Big(\boldsymbol{f}(\boldsymbol{g},\boldsymbol{\psi}),\boldsymbol{\varphi}(\boldsymbol{g},\boldsymbol{\psi})\Big)\Delta\left(\frac{\boldsymbol{f},\boldsymbol{\varphi}}{\boldsymbol{g},\boldsymbol{\psi}}\right) D\boldsymbol{g}D\boldsymbol{\psi} \tag{B.29}$$

This equality has several far-reaching consequences. In particular, it shows that, in order to perform the superintegral of a Grassmann analytic function, it is not needed to go through the sometimes long and tedious expansions (B.20) and (B.19) to express this function in terms of the previous commuting and anticommuting variables $\boldsymbol{g}$ and $\boldsymbol{\psi}$. Instead, provided the Berezinian is correctly taken into account, one can rather integrate the new anticommuting variables $\boldsymbol{\varphi}$ according to Berezin's rule, and perform the remaining integrals over the commuting variables $\boldsymbol{f}$ as if these variables were complex numbers.

Hence, Berezin's integration rules do not depend on the system of generators χ. In practice, one does not need to constantly refer to the initial system of generators and one can rather consider the anticommuting variables at each step of the calculations as new generators.

For the formula (B.29) to hold, it is crucial that F is compactly supported. If F is not compactly supported, some *boundary terms* must be added to the right-hand side of (B.29). However, for a change of variables $(f, \varphi) \to (g, \psi)$ with $f(g, \psi) = f(g, 0)$, these boundary terms vanish and the validity of (B.29) is restored. A more detailed account on these terms can be found in [8].

B.2.3 Gaussian Superintegrals

A particularly important type of superintegrals are the Gaussian superintegrals. Let X be a n-dimensional $\mathbb{C}$-linear space, $V \in (X \oplus X)(\Lambda)$ and $A \in L(X|X)$ be as in (B.6) and (B.9) respectively. Suppose that all the eigenvalues of mA_{BB} are positive and that mA_{FF} is not singular. Then

$$\int e^{-V^\dagger AV} DV_B DV_F = \mathrm{sdet}^{-1} A. \tag{B.30}$$

In particular, if it exists $M \in L(X|X)$ with $\det M_{FF} \neq 0$ and such that $D \equiv M^{-1}AM$ is block diagonal and $D_{BB} = D_{FF}$, then the integral (B.30) gives 1. In this case, the integrand is said *supersymmetric*.

Appendix C

Derivatives of Determinants

The derivatives of determinants of some particular kind play a key role in our method. In general, they appear as the way to get the quantities of interest, as moments or autocorrelation functions, from some generating functions. We group here the rules governing these derivatives. A recurrent object in these formulae will be the ρ factor

$$\rho_\alpha(\sigma) \equiv \alpha^{\text{number of cycles in } \sigma} \tag{C.1}$$

defined for any $\alpha \in \mathbb{R}$ and any permutation σ. The main result is the next theorem.

Theorem C.1 *Let A be an $n \times n$ invertible complex matrix, and for $s \in \mathbb{N}$, let $A^{(1)}, \dots A^{(s)}$ be s complex $n \times n$ matrices. Then, for $\alpha \in \mathbb{R}$,*

$$\frac{\partial^s}{\partial j_1 \dots \partial j_s} \frac{\det{}^\alpha A}{\det{}^\alpha (A - j \cdot A)}\bigg|_{j=0} = \sum_{\sigma \in S_s} \rho_\alpha(\sigma) \prod_{i=1}^{s} \sum_{x_i=1}^{n} \left(A^{(i)} A^{-1}\right)_{x_i, x_{\sigma(i)}},$$

where j and A are the s-dimensional vectors containing $j_1, \dots, j_s$ and the matrices $A^{(1)}, \dots, A^{(s)}$ respectively.

Proof. Let $b_1, b'_1, \dots b_s, b'_s \in \mathbb{N}_n$. One first shows the result for the particular choice $A^{(i)} = E^{b_i, b'_i}$ where $E^{b_i, b'_i}_{x_i, x'_i} = \delta_{x_i, b_i} \delta_{x'_i, b'_i}$, that is

$$\frac{\partial^s}{\partial j_1 \dots \partial j_s} \frac{\det{}^\alpha A}{\det{}^\alpha (A - j \cdot E)}\bigg|_{j=0} = \sum_{\sigma \in S_s} \rho_\alpha(\sigma) \prod_{i=1}^{s} A^{-1}{}_{b'_i, b_{\sigma(i)}}, \tag{C.2}$$

where E denotes the s-dimensional vector formed with $E^{b_1,b_1'},\ldots,E^{b_s,b_s'}$. Then, writing

$$A^{(i)} = \sum_{x_i,x_i'=1}^{n} A^{(i)}_{x_i,x_i'} E^{x_i,x_i'}, \tag{C.3}$$

performing the change of variables

$$\frac{\partial}{\partial j_i} f(j_i A^{(i)})\Big|_{j_i=0} = \sum_{x_i,x_i'=1}^{n} A^{(i)}_{x_i,x_i'} \frac{\partial}{\partial (j_i A^{(i)}_{x_i,x_i'})} f(j_i A^{(i)})\Big|_{j_i A^{(i)}_{x_i,x_i'}=0} \tag{C.4}$$

in the left-hand side of the expression in the theorem and applying (C.2) brings us to the general case.

The proof of (C.2) goes by induction over s. For $s = 1$, the left-hand side of (C.2) reads

$$\begin{aligned}
&\frac{\partial}{\partial j_1} \det{}^{-\alpha}\left(1 - j_1 A^{-1} E^{b_1,b_1'}\right) \\
= \;& \frac{\partial}{\partial j_1} \exp\left(-\alpha \operatorname{tr} \log\left(1 - j_1 A^{-1} E^{b_1,b_1'}\right)\right) \\
= \;& \alpha \det{}^{-\alpha}\left(1 - j_1 A^{-1} E^{b_1,b_1'}\right) \frac{\partial}{\partial j_1} \operatorname{tr} \sum_{r=1}^{\infty} \frac{1}{r} \left(j_1 A^{-1} E^{b_1,b_1'}\right)^r \\
\to \;& \alpha \operatorname{tr} A^{-1} E^{b_1,b_1'} = \alpha A^{-1}{}_{b_1',b_1}
\end{aligned} \tag{C.5}$$

as j_1 tends to zero. This result is in accordance with the right-hand side of (C.2) since the unique element of S_1 is the identity permutation which contains exactly one cycle.

Now, let $s \in \mathbb{N}$, $s \geq 2$, and suppose the statement valid for $s-1$. Let $j = (j_1,\ldots,j_{s-1})$ and $E = (E^{\beta_1,\beta_1'},\ldots,E^{\beta_{s-1},\beta_{s-1}'})$. Then, one has by assumption

$$\begin{aligned}
&\frac{\partial}{\partial j_{s-1}} \cdots \frac{\partial}{\partial j_1} \frac{\det{}^{\alpha} A}{\det{}^{\alpha}(A - j_s E^{b_s,b_s'} - j \cdot E)}\Big|_{j=0} \\
= \;& \frac{\det{}^{\alpha} A}{\det{}^{\alpha}(A - j_s E^{b_s,b_s'})} \sum_{\sigma \in S_{s-1}} \rho_\alpha(\sigma) \left(A - j_s E^{b_s,b_s'}\right)^{-1}{}_{b_1',b_{\sigma(1)}} \\
& \ldots \left(A - j_s E^{b_s,b_s'}\right)^{-1}{}_{b_{s-1}',b_{\sigma(s-1)}}
\end{aligned} \tag{C.6}$$

since, for j_s small enough, the matrix $A - j_s E^{b_s,b_s'}$ is still invertible. Taking the derivative with respect to j_s on the first factor of the right-hand side and

evaluating at $j_s = 0$ gives , by (C.5),

$$\begin{aligned} & \alpha A^{-1}{}_{b'_s,b_s} \sum_{\sigma\in\mathcal{S}_{s-1}} \rho_\alpha(\sigma) A^{-1}{}_{b'_1,b_{\sigma(1)}} \cdots A^{-1}{}_{b'_{s-1},b_{\sigma(s-1)}} \\ = & \sum_{\substack{\sigma\in\mathcal{S}_s \\ \sigma(s)=s}} \rho_\alpha(\sigma) A^{-1}{}_{b'_1,b_{\sigma(1)}} \cdots A^{-1}{}_{b'_s,b_{\sigma(s)}} \end{aligned} \tag{C.7}$$

In order to get the derivative on the permutation factor in (C.6), one first needs to calculate

$$\begin{aligned} 0 &= \frac{\partial}{\partial j}\left(A - jE^{b,b'}\right)^{-1}\left(A - jE^{b,b'}\right)\Big|_{j=0} \\ &= \frac{\partial}{\partial j}\left(A - jE^{b,b'}\right)^{-1}\Big|_{j=0} A - A^{-1}E^{b,b'}, \end{aligned} \tag{C.8}$$

which implies that

$$\begin{aligned} \frac{\partial}{\partial j}\left(A - jE^{b,b'}\right)^{-1}{}_{b'',b'''}\Big|_{j_q=0} &= \left(A^{-1}E^{b,b'}A^{-1}\right)_{b'',b'''} \\ &= A^{-1}{}_{b'',b}A^{-1}{}_{b',b'''} \end{aligned} \tag{C.9}$$

Making use of this formula, taking the derivative with respect to j_s on the permutation factor in (C.6) and evaluating at $j_q = 0$ yields

$$\begin{aligned} & \textstyle\sum_{\sigma\in\mathcal{S}_{s-1}} \rho_\alpha(\sigma) \sum_{j=1}^{s-1} A^{-1}{}_{b'_1,b_{\sigma(1)}} \cdots A^{-1}{}_{b'_{j-1},b_{\sigma(j-1)}} A^{-1}{}_{b'_j,b_s} \\ & A^{-1}{}_{b'_{j+1},b_{\sigma(j+1)}} \cdots A^{-1}{}_{b'_{s-1},b_{\sigma(s-1)}} A^{-1}{}_{b'_s,b_{\sigma(j)}} \end{aligned} \tag{C.10}$$

Now, for each pair (σ, j) with $\sigma \in \mathcal{S}_{s-1}$ and $j \in \mathbb{N}_{s-1}$, define the permutation $\tau_{\sigma,j} \in \mathcal{S}_s$ in the following way:

$$\tau_{\sigma,j} : \begin{pmatrix} 1 & \cdots & j-1 & j & j+1 & \cdots & s-1 & s \\ \sigma(1) & \cdots & \sigma(j-1) & s & \sigma(j+1) & \cdots & \sigma(s-1) & \sigma(j) \end{pmatrix} \tag{C.11}$$

In order to obtain $\tau_{\sigma,j}$ from σ, one breaks the arrow $j \longrightarrow \sigma(j)$ and replaces it by $j \longrightarrow s$ and $s \longrightarrow \sigma(j)$, that is, one inserts the new element s between j and $\sigma(j)$. Notice that $\tau_{\sigma,j}$ is indeed a permutation in $\mathcal{S}_s$ and that the columns in (C.11) are exactly the components indices of the matrices A^{-1} appearing in the term (σ, j) of (C.10). It is easy to convince oneself that $\tau_{\sigma,j} \neq \tau_{\sigma',j'}$, unless

$(\sigma, j) = (\sigma', j')$. Thus, the set $\{\tau_{\sigma,j} : \sigma \in S_{s-1}, j \in \mathbb{N}_{s-1}\}$ has $(s-1)(s-1)!$ elements, and $\tau_{\sigma,j}(s) = s$ is never realized. Besides, the set $\{\sigma \in S_s : \sigma(s) \neq s\}$ also contains $(s-1)(s-1)!$ elements. Therefore, one deduces that these two sets are the same. Moreover, σ and $\tau_{\sigma,j}$ have same number of cycles, and thus $\rho_\alpha(\sigma) = \rho_\alpha(\tau_{\sigma,j})$. With all these remarks, (C.10) becomes

$$\sum_{\substack{\sigma \in S_s \\ \sigma(s) \neq s}} \rho_\alpha(\sigma) {A^{-1}}_{b'_1, b_{\sigma(1)}} \cdots {A^{-1}}_{b'_s, b_{\sigma(s)}}. \tag{C.12}$$

Adding this contribution to (C.7) terminates the proof of (C.2) by induction.

□

In the proof of the previous theorem, the next important particular case has been shown.

Corollary C.2 *Let A and j be as in the theorem, and let E be the vector of matrices $(E^{b_1,b'_1}, \ldots, E^{b_s,b'_s})$ for some $b_1, b'_1, \ldots, b_s, b'_s \in \mathbb{N}_n$, where $E^{b,b'}$ denotes the $n \times n$ matrix having one as component (b, b') and zero everywhere else. Then*

$$\frac{\partial^s}{\partial j_1 \ldots \partial j_s} \frac{\det^\alpha A}{\det^\alpha (A - j \cdot E)}\bigg|_{j=0} = \sum_{\sigma \in S_s} \rho_\alpha(\sigma) \prod_{i=1}^{s} {A^{-1}}_{b'_i, b_{\sigma(i)}}.$$

Let us also mention the following interesting result asserting that the ρ factor can be seen as a quantity generalizing the signature of a permutation.

Theorem C.3 *Let $\sigma \in S_s$. Its signature $(-1)^\sigma$ is given by*

$$(-1)^\sigma = (-1)^s \rho_{-1}(\sigma).$$

Proof. Using the same notations as in Corollary C.2, we will show by induction on s that

$$\frac{\partial^s}{\partial j_1 \ldots \partial j_s} \frac{\det(A - j \cdot E)}{\det A}\bigg|_{j=0} = (-1)^s \sum_{\sigma \in S_s} (-1)^\sigma \prod_{i=1}^{s} {A^{-1}}_{b'_i, b_{\sigma(i)}}. \tag{C.13}$$

A direct comparison with Corollary C.2 then terminates the proof. The case $s = 1$ has already been checked in the proof of Theorem C.1. Indeed, it is

sufficient to set $\alpha = -1$ in (C.5). Similarly to (C.6), one writes the induction assumption

$$\begin{aligned}
&\frac{\partial}{\partial j_{s-1}} \cdots \frac{\partial}{\partial j_1} \left. \frac{\det\left(A - j_s E^{b_s,b'_s} - \boldsymbol{j}\cdot\boldsymbol{E}\right)}{\det A} \right|_{\mathrm{j}=0} \\
&= \frac{\det\left(A - j_s E^{b_s,b'_s}\right)}{\det A} (-1)^{s-1} \sum_{\sigma\in\mathcal{S}_{s-1}} (-1)^{\sigma} \left(A - j_s E^{b_s,b'_s}\right)^{-1}{}_{b'_1,b_{\sigma(1)}} \\
&\quad \ldots \left(A - j_s E^{b_s,b'_s}\right)^{-1}{}_{b'_{s-1},b_{\sigma(s-1)}}
\end{aligned} \tag{C.14}$$

One has now to take the derivative with respect to j_s on the expression and evaluate at $j_s = 0$. If this derivative is performed on the ratio of determinants in the right-hand side of (C.14), one finds as in (C.7)

$$\begin{aligned}
&-A^{-1}{}_{b'_s,b_s} (-1)^{s-1} \sum_{\sigma\in\mathcal{S}_{s-1}} (-1)^{\sigma} A^{-1}{}_{b'_1,b_{\sigma(1)}} \cdots A^{-1}{}_{b'_{s-1},b_{\sigma(s-1)}} \\
&= (-1)^{s} \sum_{\substack{\sigma\in\mathcal{S}_s \\ \sigma(s)=s}} (-1)^{\sigma} A^{-1}{}_{b'_1,b_{\sigma(1)}} \cdots A^{-1}{}_{b'_s,b_{\sigma(s)}}
\end{aligned} \tag{C.15}$$

Similarly to (C.10), if the derivative is taken on the permutation sum of (C.14) one finds

$$\begin{aligned}
&(-1)^{s-1} \textstyle\sum_{\sigma\in\mathcal{S}_{s-1}} (-1)^{\sigma} \sum_{j=1}^{s-1} A^{-1}{}_{b'_1,b_{\sigma(1)}} \cdots A^{-1}{}_{b'_{j-1},b_{\sigma(j-1)}} A^{-1}{}_{b'_j,b_s} \\
&\qquad A^{-1}{}_{b'_{j+1},b_{\sigma(j+1)}} \cdots A^{-1}{}_{b'_{s-1},b_{\sigma(s-1)}} A^{-1}{}_{b'_s,b_{\sigma(j)}}
\end{aligned} \tag{C.16}$$

One can define the permutations $\tau_{\sigma,j} \in S_s$ as in (C.11). Let us recall that when σ runs over S_{s-1} and j runs over $\mathbb{N}_{s-1}$, $\tau_{\sigma,j} \in S_s$ runs over the set of permutations in S_s that move the element s. Let us now investigate the relation between the signature of $\sigma \in \mathcal{S}_{q-1}$ and the signature of $\tau_{\sigma,j} \in \mathcal{S}_q$. For this purpose, decompose σ as a product of cycles. Of course, j and $\sigma(j)$ belong to the same cycle, and in this cycle $\sigma(j)$ follows j. Write also $\tau_{\sigma,j}$ as a product of cycles. In this decomposition, j, s and $\sigma(j)$ are consecutive elements in one of the cycles. It is clear by the definition (C.11) of $\tau_{\sigma,j}$ that all the cycles in σ and $\tau_{\sigma,j}$ coincide apart from the one containing j and $\sigma(j)$ which contains one more elements in $\tau_{\sigma,j}$ than it does in σ. Since the signature of a k-cycle

is completely determined by its length k through the formula $(-1)^{k+1}$, one deduces that the signature of $\tau_{\sigma,j}$ is always the opposite of the signature of σ, whatever $j \in \mathbb{N}_{s-1}$ is. With all these remarks, the expression (C.16) becomes

$$(-1)^s \sum_{\substack{\sigma \in S_s \\ \sigma(s) \neq s}} (-1)^\sigma A^{-1}{}_{b_1,b_{\sigma(1)}} \dots A^{-1}{}_{b_s,b_{\sigma(s)}}. \tag{C.17}$$

Adding (C.15) to this result finishes the induction step.

□

In the case where only one source j is considered, the formula in Theorem C.1 can be generalized to determinants of nonlinear functions. This is precisely the purpose of the next result.

Theorem C.4 *For any $\alpha \in \mathbb{R}$ and any subunitary matrix $M(j)$,*

$$\frac{\partial}{\partial j} \log \det \left(1 - M(j)\right)^\alpha = -\alpha tr \left[M'(j) \frac{1}{1 - M(j)} \right].$$

Proof. If L denotes the left-hand side of the equation in the theorem, one can write

$$\begin{aligned} L &= \alpha \frac{\partial}{\partial j} \mathrm{tr} \log \left(1 - M(j)\right) \\ &= -\alpha \sum_{n=1}^{\infty} \frac{1}{n} \frac{\partial}{\partial j} \mathrm{tr} M(j)^n \\ &= -\alpha \sum_{n=1}^{\infty} \frac{1}{n} \mathrm{tr} \left[n M(j)^{n-1} M'(j) \right] \\ &= -\alpha \mathrm{tr} \left[M'(j) \frac{1}{1 - M(j)} \right]. \end{aligned} \tag{C.18}$$

The third equality is obtained by invariance of the trace under cyclic permutations, and the fourth by summing the geometric series.

□

Lemma C.5 *For any $\alpha \in \mathbb{R}$, and any integers $n, m \in \mathbb{N}_0$,*

$$\sum_{\sigma \in S_{n,m}} \rho_\alpha(\sigma) = \frac{\partial^n}{\partial j_1^n} \frac{1}{(1 - j_1)^\alpha} \bigg|_{j_1 = 0} \frac{\partial^m}{\partial j_2^m} \frac{1}{(1 - j_2)^\alpha} \bigg|_{j_2 = 0}.$$

From this lemma, one can deduce the following result

Lemma C.6 *For any integer $N \in \mathbb{N}$, the following equality holds.*

$$\sum_{n=0}^{N} \binom{N}{n} \sum_{\sigma \in S_{n,N-n}} \rho_{\frac{1}{2}}(\sigma) = N!$$

Proof of C.5. Let us start by introducing the following block-diagonal matrix,

$$M(n,m) \equiv \left(\begin{array}{c|c} \mathrm{one}(n,n) & 0 \\ \hline 0 & \mathrm{one}(m,m) \end{array} \right), \tag{C.19}$$

where $\mathrm{one}(p,q)$ denotes the $p \times q$ matrix having all entries equal to one. Let us also define the determinant

$$D(j) \equiv \det \left[1 - \sum_{k=1}^{n+m} j_k E^{k,k} M(n,m) \right] \tag{C.20}$$

depending on $n+m$ real numbers $j_1, \ldots, j_{n+m}$ forming the vector j. Working out the matrix in this determinant, it can be seen that D factorizes as $D(j) = D_1(j) D_2(j)$, where

$$D_1(j) \equiv \begin{vmatrix} 1-j_1 & -j_1 & \ldots & -j_1 \\ -j_2 & 1-j_2 & \ldots & -j_2 \\ \vdots & \vdots & & \vdots \\ -j_n & -j_n & \ldots & 1-j_n \end{vmatrix}, \tag{C.21}$$

and $D_2(j)$ is given by the same formula with $j_{n+1}, \ldots j_{n+m}$ in place of $j_1, \ldots, j_n$. In (C.22), one can add to the first row all the other rows without modifying the value of the determinant. This yields

$$D_1(j) = \left(1 - \sum_{k=1}^{n} j_k \right) \begin{vmatrix} 1 & 1 & \ldots & 1 \\ -j_2 & 1-j_2 & \ldots & -j_2 \\ \vdots & \vdots & & \vdots \\ -j_n & -j_n & \ldots & 1-j_n \end{vmatrix}. \tag{C.22}$$

The remaining determinant is then easily shown to be one. In order to see this, it is indeed sufficient to add j_k times the first row to the row number k

recursively for $k = 2, \ldots, n$. Therefore,

$$D(j) = \left(1 - \sum_{k=1}^{n} j_k\right)\left(1 - \sum_{k=n+1}^{n+m} j_k\right). \tag{C.23}$$

Let us now take the derivatives on $D(j)$. Firstly, from the equation (C.23), it is not difficult to see that

$$\frac{\partial^N}{\partial j_1 \ldots \partial j_N} D^{-\alpha}(j)\Big|_{j=0} = \frac{\partial^{n+m}}{\partial j_1^n \partial j_2^m}(1-j_1)^{-\alpha}(1-j_2)^{-\alpha}\Big|_{j_1=j_2=0}. \tag{C.24}$$

Notice that the right-hand side of this equality is precisely the right-hand side of the first equation in the lemma. One has thus to show that the left-hand sides coincide. But making use of Corollary C.2, the left-hand side of (C.24) indeed reads

$$\frac{\partial^N}{\partial j_1 \ldots \partial j_N} D^{-\alpha}(j)\Big|_{j=0} = \sum_{\sigma \in S_{n,m}} \rho_\alpha(\sigma). \tag{C.25}$$

This proves the identity in the lemma.

□

Proof of C.6. Let us define $[1] \equiv (1-j_1)^{-1}$ and $[2] \equiv (1-j_2)^{-1}$. The symbols ∂_1 and ∂_2 will denote the derivatives with respect to j_1 and j_2 respectively. One makes use of Lemma C.5 to write the left-hand side as

$$\begin{aligned}\sum_{n=0}^{N}\binom{N}{n}\sum_{\sigma\in S_{n,N-n}}\rho_{\frac{1}{2}}(\sigma) &= \sum_{n=0}^{N}\binom{N}{n}\partial_1{}^n\partial_2{}^{N-n}\,[1]^{1/2}[2]^{1/2}\Big|_{j=0} \\ &= (\partial_1+\partial_2)^N\,[1]^{1/2}[2]^{1/2}\Big|_{j=0}.\end{aligned} \tag{C.26}$$

The proof will now go by induction over N. As $N = 1$, the result follows from (C.26) and the trivial identity $1/2 + 1/2 = 1!$. Now, for each $N \in \mathbb{N}$, let us introduce the function of j_1 and j_2

$$T_N(j) \equiv (\partial_1 + \partial_2)^N\,[1]^{1/2}[2]^{1/2}. \tag{C.27}$$

Notice that T_N can also be defined by the one step inductive process $T_{N+1} = (\partial_1 + \partial_2)T_N$ with initial condition $T_0 = [1]^{1/2}[2]^{1/2}$. With this notation, the identity (C.26) will prove the lemma once the recursive relation

$$T_{N+1}(0) = (\partial_1 + \partial_2)T_N(0) = (N+1)T_N(0) \tag{C.28}$$

is shown to hold. In order to prove (C.28), the main idea is to consider the function $T_N(j)$ as the sum of 2^N contributions of the type

$$t[x] \equiv \partial_{x(N)} \dots \partial_{x(1)} [1]^{1/2} [2]^{1/2}, \tag{C.29}$$

where $x = (x(1), \dots, x(N))$ is a sequence in $\{1, 2\}$. It is interesting to notice that the process (C.27) to obtain $T_N(j)$ by letting N grow can naturally be seen as a finite and rooted 2-regular tree, in which each end-vertex is at a topological distance N from the root. In this tree picture, the contributions $t[x]$ of $T_N(j)$ are precisely the end-vertices, and, as the operator $\partial_1 + \partial_2$ is applied, each of them gives rise to two different neighboring contributions $t[x, 1]$ and $t[x, 2]$ to the next tree $T_{N+1}(j)$. Since all the terms in $T_{N+1}(j)$ can be grouped in this way in pairs of neighboring contributions arising from the previous tree $T_N(j)$, it is sufficient to show that

$$t[x, 1]\big|_{j=0} + t[x, 2]\big|_{j=0} = (N + 1)\, t[x]\big|_{j=0} \tag{C.30}$$

in order to prove the recursive relation (C.28). Now, for any sequence x in $\{1, 2\}^N$, let us define the real numbers $s_1[x]$ and $s_2[x]$ by the relation

$$t[x] = t[x]\big|_{j=0} [1]^{s_1[x]} [2]^{s_2[x]}. \tag{C.31}$$

It is easy to convince oneself that $s_1[x] + s_2[x] = N + 1$. Therefore,

$$\begin{aligned} t[x, 1]\big|_{j=0} + t[x, 2]\big|_{j=0} &= (\partial_1 + \partial_2)\, t[x]\big|_{j=0} \\ &= (s_1[x] + s_2[x])\, t[x]\big|_{j=0} \\ &= (N + 1)\, t[x]\big|_{j=0}, \end{aligned} \tag{C.32}$$

which proves the induction step (C.30), and hence the lemma.

□

Appendix D

Determinants of Block Matrices

In this appendix, some useful rules to compute determinants of block matrices are presented. We first give the result in the commutative case.

Theorem D.1 *Let F be a commutative ring and let R be a commutative subring of $Mat(n, F)$. If $M \in Mat(m, R)$, M can also be seen as an element of $Mat(mn, F)$ and*

$$det_F M = det_F det_R M.$$

The non-trivial part is of course when the subring $R \subset \mathrm{Mat}(n, F)$ is non-commutative. Let us first give an example. Consider the matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{D.1}$$

where $A, B, C, D \in \mathrm{Mat}(n, F)$. First, if the matrices A, B, C, D commute with each other, then it exists a commutative subring R of $\mathrm{Mat}(n, F)$ containing A, B, C, D, and Theorem D claims that

$$\det_F M = \det_F(AD - BC). \tag{D.2}$$

However, if this commutation property is not fulfilled, then any subring R of $\mathrm{Mat}(n, F)$ containing A, B, C, D is not commutative, and $\det_R M$ is ill-defined. In this example, $AD - BC$ will in general differ from $DA - BC$ and $DA - CB$, and we can expect these matrices to have different determinants over F.

We now give an analog of this theorem in the noncommutative case for $m = 2$. One can directly consider the most general case where the commutative ring F is replaced with the Grassmann algebra Λ.

Theorem D.2 *Let $R = Mat(\mathbb{C}^n|\mathbb{C}^n)$ and consider the supermatrix*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

in $Mat(2, R)$ having invertible blocks A and D. Then,

$$\begin{aligned} sdet\, M &= sdet(AD)sdet(1 - A^{-1}BD^{-1}C) \\ &= sdet(AD)sdet(1 - D^{-1}CA^{-1}B). \end{aligned}$$

Proof. Decompose the supermatrix M in the theorem as the product

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 1 & A^{-1}B \\ D^{-1}C & 1 \end{pmatrix}. \tag{D.3}$$

Taking the determinant on both sides of this equation and applying the next lemma finishes the proof.

□

Lemma D.3 *Suppose that M is as in the theorem with $A = D = 1$. Then*

$$sdet\, M = sdet(1 - BC) = sdet(1 - CB).$$

Proof. Start with the identity

$$\mathrm{sdet} M = \exp \mathrm{str} \log \left[1 + \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \right]. \tag{D.4}$$

Then, expanding the logarithm as a power series yields

$$\begin{aligned} \mathrm{sdet} M &= \exp \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \mathrm{str} \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}^n \\ &= \exp \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{2n} \mathrm{str} \begin{pmatrix} (BC)^n & 0 \\ 0 & (CB)^n \end{pmatrix}, \end{aligned} \tag{D.5}$$

where we have used the fact that the odd powers of $M - 1$ are traceless. In the last forumla, the supertrace is equal to $2\mathrm{str}(BC)^n$ and $2\mathrm{str}(CB)^n$. Using one of these expression and then the other leads to the two equalities in the lemma.

□

Appendix E

Variation of the Action

E.1 Supertrace Functionals

The goal of this appendix is to obtain and solve the saddle-point equations induced by the source-free action

$$\begin{aligned} \mathcal{S}_0[Z,\tilde{Z}] &= -\mathrm{str}\log(1-Z\tilde{Z}) + \frac{1}{2}\mathrm{str}\log(1-Z\mathcal{S}_\epsilon^\dagger Z^\tau \mathcal{S}_\epsilon) \\ &\quad + \frac{1}{2}\mathrm{str}\log\left(1-\tilde{Z}^\tau\tilde{Z}\right). \end{aligned} \tag{E.1}$$

One first needs to define what is meant by variation of the action. For any functional $\rho : L(\mathbb{C}^n|\mathbb{C}^n) \to \Lambda$, the derivative of ρ in the direction $W \in L(\mathbb{C}^n|\mathbb{C}^n)$ is defined by

$$D_W\rho(Z) \equiv \lim_{\epsilon\to 0}\frac{\rho(Z+\epsilon W)-\rho(Z)}{\epsilon}. \tag{E.2}$$

An important case for our purpose is when ρ is the supertrace of some matrix map. In this situation, one can give a simple formula for its directional derivative.

Lemma E.1 *Let f be an analytic function on an open domain $\Omega \subset \mathbb{C}$. We also denote by f the mapping it induces on the elements of $L(\mathbb{C}^n|\mathbb{C}^n)$ whose bodies have eigenvalues in Ω. Let ρ be the functional*

$$\rho(Z) = str\Big[f\big(AZBZ\big)\Big], \quad \textit{where} \quad A, B \in L(\mathbb{C}^n|\mathbb{C}^n).$$

Then, the derivative of ρ in the direction W is

$$D_W\rho(Z) = str\Big[f'(AZBZ)\big(AWBZ + AZBW\big)\Big].$$

This lemma can easily be generalized to $\rho(Z) = \mathrm{str}[f(A_1ZA_2Z\ldots A_kZ]$. The derivative of such a functional ρ is obtained by taking the sum of derivatives with respect to each factor Z separately.

Proof. One can always suppose that Ω is a neighborhood of the origin in the complex plane, since otherwise this situation is achieved by shifting the variable. By the definition (E.2), one has

$$D_W\rho(Z) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\mathrm{str}\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\Big(\big[A(Z+\epsilon W)B(Z+\epsilon W)\big]^n - \big[AZBZ\big]^n\Big), \quad \text{(E.3)}$$

Now comes the simplification due to the supertrace. Indeed, observe that at first order in ϵ,

$$\begin{aligned} & \mathrm{str}\big[A(Z+\epsilon W)B(Z+\epsilon W)\big]^n \\ = \ & \mathrm{str}\big[AZBZ\big]^n + \epsilon\cdot\mathrm{str}\big[n(AZBZ)^{n-1}AWBZ\big] \\ & +\epsilon\cdot\mathrm{str}\big[n(AZBZ)^{n-1}AZBW\big] \end{aligned} \quad \text{(E.4)}$$

Injecting this result into (E.3) terminates the proof.

□

Examples We now give three examples of use of this lemma that will directly lead to the variation of the source-free action S_0.

1. Consider the functional $\rho_1[Z] = -\mathrm{str}\log(1 - Z\tilde{Z})$. By the lemma

$$D_W\rho_1[Z] = \mathrm{str}\Big((1 - Z\tilde{Z})^{-1}W\tilde{Z}\Big). \quad \text{(E.5)}$$

Similarly, if $\rho_1'[\tilde{Z}] = -\mathrm{str}\log(1 - Z\tilde{Z})$, then

$$D_W\rho_1'[\tilde{Z}] = \mathrm{str}\Big((1 - Z\tilde{Z})^{-1}ZW\Big) \quad \text{(E.6)}$$

2. Consider the functional $\rho_2[Z] = \mathrm{str}\log(1 - Z\mathcal{S}_\epsilon^\dagger Z^\tau \mathcal{S}_\epsilon)$. The lemma implies that

$$D_W\rho_2[Z] = \Big(D_{1,W}\tilde{\rho}_2[Z_1, Z_2] + D_{2,W}\tilde{\rho}_2[Z_1, Z_2]\Big|_{Z_1=Z_2=Z}, \tag{E.7}$$

where $\tilde{\rho}_2[Z_1, Z_2] = \mathrm{str}\log(1 - Z_1\mathcal{S}_\epsilon^\dagger Z_2^\tau \mathcal{S}_\epsilon)$, and $D_{i,W}$ is the derivative with respect to Z_i in the direction W. The first term yields

$$D_{1,W}\tilde{\rho}_2[Z_1, Z_2] = -\mathrm{str}\Big[(1 - Z_1\mathcal{S}_\epsilon^\dagger Z_2^\tau \mathcal{S}_\epsilon)^{-1} W \mathcal{S}_\epsilon^\dagger Z_2^\tau \mathcal{S}_\epsilon\Big]. \tag{E.8}$$

To compute the second term in (E.7), we use the fact that the supertrace of the generalized transpose of a matrix is equal to the supertrace of this matrix. Since $\mathcal{S}_\epsilon^\tau = \mathcal{S}_\epsilon$, this second term thus becomes

$$\begin{aligned} D_{2,W}\tilde{\rho}_2[Z_1, Z_2] &= D_{2,W}\mathrm{str}\log\left(1 - \mathcal{S}_\epsilon Z_2 \mathcal{S}_\epsilon^\dagger Z_1^\tau\right) \\ &= -\mathrm{str}\Big[(1 - \mathcal{S}_\epsilon Z_2 \mathcal{S}_\epsilon^\dagger Z_1^\tau)^{-1} \mathcal{S}_\epsilon W \mathcal{S}_\epsilon^\dagger Z_1^\tau\Big]. \end{aligned} \tag{E.9}$$

And finally, using the invariance of the supertrace under cyclic permutations, one obtains

$$D_W\rho_2[Z] = -2\,\mathrm{str}\Big[\left(1 - \mathcal{S}_\epsilon^\dagger Z^\tau \mathcal{S}_\epsilon Z\right)^{-1} \mathcal{S}_\epsilon^\dagger Z^\tau \mathcal{S}_\epsilon W\Big]. \tag{E.10}$$

3. Consider the functional $\rho_3[\tilde{Z}] = \mathrm{str}\log(1 - \tilde{Z}^\tau \tilde{Z})$. Using the same method as before,

$$D_W\rho_3[\tilde{Z}] = \Big(D_{1,W}\tilde{\rho}_3[\tilde{Z}_1, \tilde{Z}_2] + D_{2,W}\tilde{\rho}_3[\tilde{Z}_1, \tilde{Z}_2]\Big|_{\tilde{Z}_1=\tilde{Z}_2=\tilde{Z}}, \tag{E.11}$$

where $\tilde{\rho}_3[\tilde{Z}_1, \tilde{Z}_2] = \mathrm{str}\log(1 - \tilde{Z}_1^\tau \tilde{Z}_2)$. The second term can immediately be computed and reads

$$D_{2,W}\tilde{\rho}_3[\tilde{Z}_1, \tilde{Z}_2] = -\mathrm{str}\left((1 - \tilde{Z}_1^\tau \tilde{Z}_2)^{-1} \tilde{Z}_1^\tau W\right). \tag{E.12}$$

For the first, we take the generalized transpose of the argument in the logarithm, and then we use the invariance of the supertrace under cyclic permutations. This yields

$$\begin{aligned} D_{1,W}\tilde{\rho}_3[\tilde{Z}_1, \tilde{Z}_2] &= D_{1,W}\mathrm{str}\log\left(1 - \tilde{Z}_2^\tau \tilde{Z}_1\right) \\ &= -\mathrm{str}\Big[(1 - \tilde{Z}_2^\tau \tilde{Z}_1)^{-1} \tilde{Z}_2^\tau W\Big]. \end{aligned} \tag{E.13}$$

Therefore

$$D_W\rho_3[\tilde{Z}] = -2\,\mathrm{str}\Big[(1-\tilde{Z}^\tau\tilde{Z})^{-1}\tilde{Z}^\tau W\Big] \tag{E.14}$$

E.2 Saddle-Point Equations

The two saddle-point equations are

$$D_{1,W}\mathcal{S}_0[Z,\tilde{Z}] = \mathcal{O}(\epsilon) \quad \text{and} \quad D_{2,W}\mathcal{S}_0[Z,\tilde{Z}] = 0, \tag{E.15}$$

for all $W \in L(TR\otimes\mathcal{A}|TR\otimes\mathcal{A})$. Notice on (E.1) that they take the form

$$\begin{cases} D_W\rho_1[Z] + \frac{1}{2}D_W\rho_2[Z] = \mathcal{O}(\epsilon) \\ D_W\rho_1'[\tilde{Z}] + \frac{1}{2}D_W\rho_3[\tilde{Z}] = 0 \end{cases} \tag{E.16}$$

where the ρ_i functionals are those defined in the previous examples. Using the results of these examples provides

$$\begin{cases} \mathrm{str}\Big[(1-\tilde{Z}Z)^{-1}\tilde{Z} - (1-\mathcal{S}_\epsilon^\dagger Z^\tau\mathcal{S}_\epsilon Z)^{-1}\mathcal{S}_\epsilon^\dagger Z^\tau\mathcal{S}_\epsilon\Big]W = \mathcal{O}(\epsilon) \\ \mathrm{str}\Big[(1-Z\tilde{Z})^{-1}Z - (1-\tilde{Z}^\tau\tilde{Z})^{-1}\tilde{Z}^\tau\Big]W = 0 \end{cases} \tag{E.17}$$

Since these equations must be satisfied for all W, the supermatrices in the square brackets must vanish. The second equation leads to $Z = \tilde{Z}^\tau$, or

$$\tilde{Z} = Z^\tau. \tag{E.18}$$

Making use of this relation, the first equation of (E.17) implies that Z and $\mathcal{S}$ must commute with each other. The requirement $\mathcal{S}Z - Z\mathcal{S} = 0$ can be explicitly written in time-reversal space as

$$\begin{pmatrix} SZ_{\uparrow\uparrow} - Z_{\uparrow\uparrow}S & SZ_{\uparrow\downarrow} - Z_{\uparrow\downarrow}S^\mathcal{T} \\ S^\mathcal{T}Z_{\downarrow\uparrow} - Z_{\downarrow\uparrow}S & S^\mathcal{T}Z_{\downarrow\downarrow} - Z_{\downarrow\downarrow}S^\mathcal{T} \end{pmatrix} = 0 \tag{E.19}$$

If the graph is in the orthogonal class, $S^\mathcal{T} = S$, and thus each component of Z must commute with S. But the supermatrices Z and $\tilde{Z}$, which originate from the color-flavor transformation leading to Theorem 3.4, are diagonal in the bond space $\mathcal{A}_b$. Therefore, since the classical map $M_{\beta,\beta'} = |S_{\beta,\beta'}|^2$ is

supposed mixing, the only matrices diagonal in bond space and commuting with S must be the multiple of identity. In other terms, Z and $\tilde{Z}$ can be written

$$Z = \mathbb{1}_{\mathcal{A}} \otimes Y \quad \text{and} \quad \tilde{Z} = \mathbb{1}_{\mathcal{A}} \otimes \tilde{Y} \tag{E.20}$$

for some supermatrices $Y, \tilde{Y} \in L(TR|TR)$ satisfying the condition

$$\tilde{Y} = Y^{\tau}. \tag{E.21}$$

The couples of supermatrices $(Z_0, \tilde{Z}_0)$ satisfying (E.20) and (E.21) constitute the zero mode, or the mean field mode, of the orthogonal symmetry class.

If the graph is in the unitary class, the condition $S = S^{\mathcal{T}}$ must be released. The commutation requirement (E.19) then becomes a stronger condition than before, and the set of supermatrices Z satisfying this requirement is a subset of the supermatrices Z found in (E.20). The equations in the diagonal components $(\uparrow,\uparrow)$ and $(\downarrow,\downarrow)$ of (E.19) remain unchanged, but the off-diagonal ones become

$$\begin{cases} (S - S^{\mathcal{T}})\, Z_{\uparrow\downarrow} = 0 \\ (S - S^{\mathcal{T}})\, Z_{\downarrow\uparrow} = 0 \end{cases} \tag{E.22}$$

Since, in these equations, the first factor does not vanish, the second factors must be zero. Hence, in the unitary symmetry class, the zero modes are the couples $(Z, \tilde{Z})$ satisfying (E.20), (E.21), and the additional condition to be diagonal in time-reversal space.

Appendix F

The Q Supermatrices

F.1 Description of the Efetov's σ Model

In Section 4, the Q matrices are defined from the zero mode $(Y, \tilde{Y})$ which lives in $L(TR|TR)$. We give here some information about the geometry of the Q matrices and describe the so-called Efetov's σ-model. This short account is mainly a summary of what can be found in [76].

Let us first suppose that $TR \simeq \mathbb{C}^1$, which means that no time-reversal doubling has been performed, and let us consider the Lie supergroup

$$G \equiv L(TR \otimes RA|TR \otimes RA), \tag{F.1}$$

Let also H be the centralizer of σ_3^{RA} in G. This subgroup consists of the supermatrices of the form

$$h = \begin{pmatrix} h_r & 0 \\ 0 & h_a \end{pmatrix}_{RA}. \tag{F.2}$$

The elements of G moving σ_3^{RA} form the coset space G/H. The body of G/H, that is the set found as all the Grassmann variables are set to zero, reads

$$\begin{aligned} m(G/H) \equiv & \underbrace{L(TR \otimes RA)/L(TR) \times L(TR)}_{BB \text{ sector}} \\ & \oplus \underbrace{L(TR \otimes RA)/L(TR) \times L(TR)}_{FF \text{ sector}} \end{aligned} \tag{F.3}$$

Besides, the elements of the coset space G/H are in one-to-one relation with the supermatrices $Q = g\sigma_3^{RA}g^{-1}$. In the Efetov's σ model with unitary symmetry, one is interested in integrating Grassmann analytic functions $f(Q)$ over the domain $m(Q) \in M_B \times M_F$ in $m(G/H)$, where

$$M_B \equiv U(TR,TR)/U(TR) \times U(TR), \tag{F.4}$$

$$M_F \equiv U(TR \otimes RA)/U(TR) \times U(TR). \tag{F.5}$$

Notice that, similarly to (F.3), the product $U(TR) \times U(TR)$ is the centralizer of σ_3^{RA} in both the pseudounitary group $U(TR,TR)$ and the unitary group $U(TR \otimes RA)$, so that M_B (resp. M_F) is the set of pseudounitary (resp. unitary) matrices moving σ_3^{RA} in $U(TR,TR)$ (resp. $U(TR \otimes RA)$). This remark implies that $M_B \times M_F$ is indeed a subgroup of $m(G/H)$.

The integration measure DQ used in the Efetov's σ model is the G-invariant measure, so that for all $g_0 \in G$,

$$\int_{m(G/H)} DQ f(Q) = \int_{m(G/H)} DQ f(g_0 Q). \tag{F.6}$$

As explained in [76], a shift by $g_0 \in G$ may not leave the integration domain $M_B \times M_F$ unchanged. But if the function f is Grassmann analytic, the domain can be deformed back to $M_B \times M_F$. Hence, the equality (F.6) with $M_B \times M_F$ in place of $m(G/H)$ still holds in this case. An usual technique to find out the explicit formula for the measure DQ in any set of coordinates is to use the invariance under G of the quadratic form $dQ \mapsto \mathrm{str}\,(dQ)^2$ on $\mathrm{Lie}(G)$. One indeed has

$$\begin{aligned} \mathrm{str}\,(dQ)^2 &= \mathrm{str}\left(dg\sigma_3^{RA}g^{-1} - g\sigma_3^{RA}g^{-1}dgg^{-1}\right)^2 \\ &= \mathrm{str}\left(\left[g^{-1}dg, \sigma_3^{RA}\right]^2\right), \end{aligned} \tag{F.7}$$

and the invariance of this quadratic form follows from the invariance of $g^{-1}dg$ under the shift $g \mapsto g_0 g$ for any $g_0 \in G$.

Let us just introduce some convenient notations. For any supermatrix A having a retarded-advanced structure, one defines

$$\bar{A} \equiv \mathcal{K}A^{\dagger}\mathcal{K}, \tag{F.8}$$

where $\mathcal{K}$ is the supermatrix

$$\mathcal{K} \equiv \begin{pmatrix} \sigma_3^{RA} & 0 \\ 0 & 1 \end{pmatrix}_{BF} = \begin{pmatrix} 1 & 0 \\ 0 & -\sigma_3^{BF} \end{pmatrix}_{RA} . \tag{F.9}$$

It is easy to see that $\bar{\bar{A}} = A$ and $\overline{AB} = \bar{B}\bar{A}$. Moreover, if A also possesses a structure in time-reversal space, then $\overline{A^\tau} = \bar{A}^\tau$. Besides, it follows from these definitions that a supermatrix A satisfying the equation $\bar{A}A = 1$ has a body $m(A)$ that is pseudounitary in the BB sector and unitary in the FF sector. One often just says that A is pseudounitary, remembering that the signs defining this pseudounitarity are those given by the matrix $\mathcal{K}$ in (F.9). Notice that the *rr* block of such a matrix is unitary, whereas the *aa* block is pseudounitary. Similarly, if A satisfies $\bar{A} = A$, then its body is pseudohermitian in the BB sector, and hermitian in the FF sector. In particular, a supermatrix Q in the Efetov's σ model space satisfies $\bar{Q}Q = 1$ and is thus pseudounitary. And since by definition of Q one also has $Q^2 = 1$,

$$Q = Q^{-1} = \bar{Q}, \tag{F.10}$$

which precisely implies that Q is pseudohermitian, that is, pseudohermitian in the BB sector and hermitian in the FF sector.

Let us now turn to the relation between the zero mode $(Y, \tilde{Y})$ and the supermatrices $Q = g\sigma_3^{RA}g^{-1}$ introduced above. As already mentioned, this last formula puts the Q matrices in one-to-one correspondence with G/H. Besides, it is not difficult to check that the supermatrices

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}_{RA} \quad \text{and} \quad R = \begin{pmatrix} 1 & Y = bd^{-1} \\ \tilde{Y} = ca^{-1} & 1 \end{pmatrix}_{RA} \tag{F.11}$$

are in the same coset in G/H. Therefore, one deduces that Y and $\tilde{Y}$ parametrize the Lie supergroup G defined in (F.1) through the formula

$$Q = R\sigma_3^{RA}R^{-1}. \tag{F.12}$$

The components of this Q matrix in the retarded-advanced space are easily found to be given by the relations (4.6) - (4.9). Let us now impose some

conditions on the matrix parameters Y and $\tilde{Y}$ in order for $m(Q)$ to be in the Efetov's σ model space $M_B \times M_F$ defined in (F.4) and (F.5). Let us first impose the required unitarity in the Fermi-Fermi sector. Dropping the FF indices to keep the notations as simple as possible, one has

$$\begin{pmatrix} 1 & Y \\ \tilde{Y} & 1 \end{pmatrix}^{\dagger} \begin{pmatrix} 1 & Y \\ \tilde{Y} & 1 \end{pmatrix} = \begin{pmatrix} 1+\tilde{Y}^{\dagger}\tilde{Y} & Y+\tilde{Y}^{\dagger} \\ Y^{\dagger}+\tilde{Y} & 1+Y^{\dagger}Y \end{pmatrix} \tag{F.13}$$

which corresponds, by the procedure in (F.11), to the R_{FF} matrix

$$\begin{pmatrix} 1 & \left(Y+\tilde{Y}^{\dagger}\right)\left(1+Y^{\dagger}Y\right)^{-1} \\ \left(Y^{\dagger}+\tilde{Y}\right)\left(1+\tilde{Y}^{\dagger}\tilde{Y}\right)^{-1} & 1 \end{pmatrix}. \tag{F.14}$$

Therefore, imposing unitarity in the Fermi-Fermi sector merely amounts to require that the fields Y_{FF} and $\tilde{Y}_{FF}$ satisfy

$$\tilde{Y}_{FF} = -Y_{FF}^{\dagger}. \tag{F.15}$$

Moreover, the complex matrix $m(Y_{FF}^{\dagger}Y_{FF})$ is always positive, so that there is no further restriction for the inverse matrices occurring in (F.14) to exist. Let us now similarly impose pseudounitarity in the Bose-Bose sector. Dropping the BB indices, one has

$$\sigma_3^{RA} \begin{pmatrix} 1 & Y \\ \tilde{Y} & 1 \end{pmatrix}^{\dagger} \sigma_3^{RA} \begin{pmatrix} 1 & Y \\ \tilde{Y} & 1 \end{pmatrix} = \begin{pmatrix} 1-\tilde{Y}^{\dagger}\tilde{Y} & Y-\tilde{Y}^{\dagger} \\ \tilde{Y}-Y^{\dagger} & 1-Y^{\dagger}Y \end{pmatrix} \tag{F.16}$$

which corresponds, by the procedure in (F.11), to the R_{BB} matrix

$$\begin{pmatrix} 1 & \left(Y-\tilde{Y}^{\dagger}\right)\left(1-Y^{\dagger}Y\right)^{-1} \\ \left(\tilde{Y}-Y^{\dagger}\right)\left(1-\tilde{Y}^{\dagger}\tilde{Y}\right)^{-1} & 1 \end{pmatrix}. \tag{F.17}$$

Now, we first need to impose that the eigenvalues of the positive Hermitian matrix $m(Y_{BB}^{\dagger}Y_{BB})$ are smaller than one, and then, one also has to require

$$\tilde{Y}_{BB} = Y_{BB}^{\dagger}. \tag{F.18}$$

Together with (F.15), these requirements are exactly those of the color-flavor theorem. If no time-reversal doubling is performed, the fields Y and $\tilde{Y}$ obtained

by a saddle-point argument from the color-flavor fields Z and $\tilde{Z}$ are precisely the parameters of the Efetov σ model space with unitary symmetry.

One can now easily compute the dimension of the Efetov σ model space with unitary symmetry and with $TR \simeq \mathbb{C}^1$. If this space is parametrized by the supermatrices Y and $\tilde{Y}$, one has:

$$\left.\begin{matrix} \Re Y_{BB} & \Im Y_{BB} \\ \Re Y_{FF} & \Im Y_{FF} \end{matrix}\right\} \quad 4 \text{ real parameters} \tag{F.19}$$

$$\left.\begin{matrix} Y_{BF} & Y_{FB} \\ \tilde{Y}_{BF} & \tilde{Y}_{FB} \end{matrix}\right\} \quad 4 \text{ odd parameters} \tag{F.20}$$

Let us now consider the case where $TR \simeq \mathbb{C}^2$, which means that a time-reversal doubling has been performed. Everything remains similar to what is exposed above. The difference is that now, for the Efetov's σ model with orthogonal symmetry, the matrices Y and $\tilde{Y}$ are twice larger than before but are required to satisfy the additional constraint

$$\tilde{Y} = Y^{\intercal}. \tag{F.21}$$

From the components (4.6) - (4.9) of Q in retarded-advanced space, it is direct to check that this new constraint implies

$$Q^{\intercal} = \sigma_3^{RA} Q \sigma_3^{RA}. \tag{F.22}$$

These supermatrices Q form a subgroup of G/H where G is defined by (F.1) with $TR \simeq \mathbb{C}^2$ called orthosymplectic Lie supergroup. It is easy to see that this subgroup contains 8 real parameters and 8 anticommuting ones. More details about the Efetov's σ model with orthogonal symmetry can be found in the third section of [76]. Let us just mention that in this model, the integration domain is $M_B \times M_F$, where

$$M_B = SO(TR,TR)/SO(TR) \times SO(TR), \tag{F.23}$$

$$M_F = Sp(RA \otimes TR)/Sp(TR) \times Sp(TR). \tag{F.24}$$

These are subgroups of M_B and M_F in (F.4) and (F.5) with $TR \simeq \mathbb{C}^2$. Furthermore, the Efetov's σ model with unitary symmetry can also be treated with $TR \simeq \mathbb{C}^2$. For this purpose, one imposes to the parameter Y the additional condition to be diagonal in the TR space. By doing so, one recovers 4 real and 4 anticommuting parameters.

Let us briefly summarize the main points of this section. The parameters Y and $\tilde{Y}$ described above are precisely the solutions of the mean field equations, and thus constitute the zero mode. The equation (F.15) translates the unitarity in the FF sector, whereas (F.18) translates the pseudounitarity in the BB sector. The constraint (F.21) is due to time-reversal doubling, and the possible additional condition that Y is diagonal in time-reversal space enables to treat the unitary case with time-reversal doubling.

F.2 The Polar Coordinates

F.2.1 Without TR Doubling

It is sometimes convenient to parametrize the Efetov's σ model space using coordinates different from Y and $\tilde{Y}$. We describe here the polar coordinates introduced by K. Efetov and explained in the textbook [34]. Let us first work without time-reversal doubling, that is $TR \simeq \mathbb{C}^1$. The supermatrices Q are written as a product

$$Q = U_1 Q'_0 \bar{U}_1, \tag{F.25}$$

where the supermatrix U_1 diagonalizes Q in the Bose-Fermi space and satisfies $\bar{U}_1 U_1 = 1$. Then, it is straightforward to check that the supermatrix

$$Q'_0 = \begin{pmatrix} Q'_{0B} & 0 \\ 0 & Q'_{0F} \end{pmatrix}_{BF} \tag{F.26}$$

has to carry all the symmetries of the original supermatrix Q in the RA space, that is

$$Q'_0 = Q'^{-1}_0 = \bar{Q}'_0. \tag{F.27}$$

These two equalities can be translated into separate requirements on the BB and FF sectors, namely

$$Q'_{0B} = Q'^{-1}_{0B} = \sigma_3^{RA} Q'^{\dagger}_{0B} \sigma_3^{RA} \quad \text{and} \quad Q'_{0F} = Q'^{-1}_{0F} = Q'^{\dagger}_{0F}. \tag{F.28}$$

It is not difficult to show that the most general matrices Q'_{0B} and Q'_{0F} having these properties are

$$Q'_{0B} = \begin{pmatrix} \cosh\theta_B & -e^{i\chi}\sinh\theta_B \\ e^{-i\chi}\sinh\theta_B & -\cosh\theta_B \end{pmatrix}_{RA}, \tag{F.29}$$

for $\theta_B \in [0,\infty)$ and $\chi \in [0,2\pi)$, and

$$Q'_{0F} = \begin{pmatrix} \cos\theta_F & e^{i\phi}\sin\theta_F \\ e^{-i\phi}\sin\theta_F & -\cos\theta_F \end{pmatrix}_{RA}, \tag{F.30}$$

for $\theta_F \in [0,\pi]$ and $\phi \in [0,2\pi)$. The matrices Q'_{0B} and Q'_{0F} involve four real parameters, that is, by (F.19), the full count of real parameters of the space we want to describe here.

By the count performed in (F.20), one still needs four odd parameters. Those are contained in the pseudounitary supermatrix U_1 diagonalizing Q in Bose-Fermi space. Since Q'_0 contains the whole structure in retarded-advanced space, one can choose U_1 diagonal in this space, and one writes

$$U_1 = \begin{pmatrix} u_1 & 0 \\ 0 & v_1 \end{pmatrix}_{RA}. \tag{F.31}$$

The pseudounitarity $\bar{U}_1 U_1 = 1$ then becomes

$$\bar{u}_1 \equiv u_1^{\dagger} = u_1^{-1}, \tag{F.32}$$

$$\bar{v}_1 \equiv \sigma_3^{BF} v_1^{\dagger} \sigma_3^{BF} = v_1^{-1}. \tag{F.33}$$

Here and in the sequel, when it is clear that some supermatrices u and v belong to the rr and aa sectors respectively, we write by abuse of notation $\bar{u}$ and $\bar{v}$ for the quantities as in (F.32) and (F.33). For u_1 and v_1, one takes the

exponentials of some antihermitian and hermitian matrices respectively. For example,

$$u_1 \equiv \exp\begin{pmatrix} 0 & -2\eta^* \\ 2\eta & 0 \end{pmatrix}_{BF} = \begin{pmatrix} 1-2\eta^*\eta & -2\eta^* \\ 2\eta & 1-2\eta\eta^* \end{pmatrix}_{BF} \tag{F.34}$$

$$v_1 \equiv \exp\begin{pmatrix} 0 & -2i\kappa^* \\ 2i\kappa & 0 \end{pmatrix}_{BF} = \begin{pmatrix} 1+2\kappa^*\kappa & -2i\kappa^* \\ 2i\kappa & 1+2\kappa\kappa^* \end{pmatrix}_{BF} \tag{F.35}$$

The four odd parameters $\eta, \eta^*, \kappa, \kappa^*$ are the four odd variables of the Efetov's space.

In the textbook [34], the pseudohermitian supermatrix Q'_0 is decomposed further as

$$Q'_0 = U_2 Q_0 \bar{U}_2, \tag{F.36}$$

where the supermatrix U_2 is pseudounitary, $\bar{U}_2 U_2 = 1$, and obviously chosen diagonal in the Bose-fermi space, that is

$$U_2 = \begin{pmatrix} U_{2B} & 0 \\ 0 & U_{2F} \end{pmatrix}_{BF}, \tag{F.37}$$

with the pure blocks

$$U_{2B} = \begin{pmatrix} e^{i\xi} & 0 \\ 0 & 1 \end{pmatrix}_{RA} \quad \text{and} \quad U_{2F} = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & 1 \end{pmatrix}_{RA} \tag{F.38}$$

for angles $\xi, \phi \in [0, 2\pi)$. Defining the pseudounitary matrix $U = U_1 U_2$, the original Q matrix factorizes as

$$Q = U Q_0 \bar{U}. \tag{F.39}$$

Previously, the supermatrices Q'_0 and U_2 have first been decomposed in Bose-Fermi space, and only then in retarded-advanced space. For calculation purposes it is also convenient to decompose them in the other order. It is easy to check that the pseudohermitian matrix Q_0 can be written

$$Q_0 = \begin{pmatrix} \cos\hat{\theta} & i\sin\hat{\theta} \\ -i\sin\hat{\theta} & -\cos\hat{\theta} \end{pmatrix}_{RA}, \quad \hat{\theta} = \begin{pmatrix} i\theta_B & 0 \\ 0 & \theta_F \end{pmatrix}_{BF}. \tag{F.40}$$

for $\theta_B \in [0, \infty)$ and $\theta_F \in [0, \pi]$. We finally introduce the following notations in retarded-advanced space.

$$U \equiv U_1 U_2 = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}_{RA} = \begin{pmatrix} u_1 & 0 \\ 0 & v_1 \end{pmatrix}_{RA} \begin{pmatrix} u_2 & 0 \\ 0 & v_2 \end{pmatrix}_{RA}. \tag{F.41}$$

The supermatrices u_1 and v_1 being given by (F.34) and (F.35), this defines the supermatrices u_2 and v_2. And by comparison with (F.37), one finds

$$u_2 = \begin{pmatrix} e^{i\xi} & 0 \\ 0 & e^{i\phi} \end{pmatrix} \quad \text{and} \quad v_2 = \mathbb{1}_{BF}. \tag{F.42}$$

F.2.2 Unitary Symmetry with TR Doubling

We give here the expression of Q in terms of polar coordinates in the unitary symmetry case where a time-reversal doubling has been performed. According to the end of Section F.1, all the parameters introduced in F.2.1 will become 2×2 matrices acting on the time-reversal space. These matrices must be such that the initial symmetries $Q = Q^{-1} = \bar{Q}$ are conserved and such that Q fulfills the new identity $Q^\tau = \sigma_3^{RA} Q \sigma_3^{RA}$. Moreover, since we are here interested in the unitary case, these matrices have all to be diagonal in the TR space. These new constraints make Q have the same number of parameters as before, that is, four real parameters and four odd ones. With all these remarks, it is natural to start as before with

$$Q = U Q_0 \bar{U}, \tag{F.43}$$

where Q_0 and U have to satisfy $\bar{Q}_0 = Q_0^{-1} = Q_0$ and $\bar{U}U = 1$. We set as before

$$Q_0 = \begin{pmatrix} \cos\hat{\theta} & i\sin\hat{\theta} \\ -i\sin\hat{\theta} & -\cos\hat{\theta} \end{pmatrix}_{RA}, \quad \hat{\theta} = \begin{pmatrix} i\theta_B & 0 \\ 0 & \theta_F \end{pmatrix}_{BF}. \tag{F.44}$$

Now, θ_B and θ_F are diagonal matrices in the TR space. In order for Q_0 to remain pseudohermitian, θ_B and θ_F must be real symmetric. One also imposes

$$Q_0^\tau = \sigma_3^{RA} Q_0 \sigma_3^{RA}, \tag{F.45}$$

which forces

$$\theta_B = \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_1 \end{pmatrix}_{TR}, \quad \theta_F = \begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix}_{TR}. \tag{F.46}$$

We choose $\theta_1 \in [0, \infty)$ and $\theta \in [0, \pi]$ as previously. Q_0 thus contains two of the four real parameters. Similarly as before, the pseudounitary matrix U is chosen diagonal in the retarded-advanced space and is decomposed $U = U_1 U_2$, with U_1 and U_2 both pseudounitary. Furthermore, these two supermatrices are required to satisfy

$$U_i^\tau \stackrel{!}{=} \sigma_3^{RA} \bar{U}_i \sigma_3^{RA} = \bar{U}_i, \tag{F.47}$$

so that their product $U = U_1 U_2$ has the same property, and, together with the requirement (F.45), one has

$$Q^\tau = \left(U Q_0 \bar{U}\right)^\tau = \bar{U}^\tau Q_0^\tau U^\tau = \sigma_3^{RA} U Q_0 \bar{U} \sigma_3^{RA} = \sigma_3^{RA} Q \sigma_3^{RA} \tag{F.48}$$

as wanted. The matrices U_i are written in retarded-advanced space as previously in (F.41), that is,

$$U_i = \begin{pmatrix} u_i & 0 \\ 0 & v_i \end{pmatrix}_{RA}. \tag{F.49}$$

The component u_i must be unitary whereas v_i must be antiunitary. Moreover, the time-reversal doubling constraint (F.47) implies a constraint on u_i and another constraint for v_i. Writing explicitly these constraints in time-reversal space leads to

$$u_i^\tau \equiv \begin{pmatrix} \sigma_3^{BF} u_{i\downarrow\downarrow}^T \sigma_3^{BF} & 0 \\ 0 & u_{i\uparrow\uparrow}^T \end{pmatrix}_{TR} \stackrel{!}{=} \begin{pmatrix} u_{i\uparrow\uparrow}^\dagger & 0 \\ 0 & u_{i\downarrow\downarrow}^\dagger \end{pmatrix}_{TR} \equiv \bar{u}_i \tag{F.50}$$

and

$$v_i^\tau \equiv \begin{pmatrix} \sigma_3^{BF} v_{i\downarrow\downarrow}^T \sigma_3^{BF} & 0 \\ 0 & v_{i\uparrow\uparrow}^T \end{pmatrix}_{TR} \stackrel{!}{=} \sigma_3^{BF} \begin{pmatrix} v_{i\uparrow\uparrow}^\dagger & 0 \\ 0 & v_{i\downarrow\downarrow}^\dagger \end{pmatrix}_{TR} \sigma_3^{BF} \equiv \bar{v}_i \tag{F.51}$$

Notice that in (F.50) and (F.51), the equalities in the $\uparrow\uparrow$ components are equivalent to the equalities in the $\downarrow\downarrow$ components, so that it is sufficient to impose the latter, which are themselves equivalent to

$$u_{i\uparrow\uparrow} = u_{i\downarrow\downarrow}^* \quad \text{and} \quad v_{i\uparrow\uparrow} = \sigma_3^{BF} v_{i\downarrow\downarrow}^* \sigma_3^{BF}. \tag{F.52}$$

Let us first look at U_1, whose purpose is to diagonalize Q in Bose-Fermi space. It contains all the four anticommuting Grassman variables, and is written as in (F.34) and (F.35), that is

$$U_1 = \begin{pmatrix} u_1 & 0 \\ 0 & v_1 \end{pmatrix}_{RA}, \quad \text{with} \quad \begin{cases} u_1 = \begin{pmatrix} 1-2\eta^\dagger\eta & -2\eta^\dagger \\ 2\eta & 1-2\eta\eta^\dagger \end{pmatrix}_{BF} \\ v_1 = \begin{pmatrix} 1+2\kappa^\dagger\kappa & -2i\kappa^\dagger \\ 2i\kappa & 1+2\kappa\kappa^\dagger \end{pmatrix}_{BF} \end{cases} \tag{F.53}$$

In these equalities η and κ are 2×2 diagonal matrices acting on the TR space whose components are all anticommuting. It follows that $\eta^{\dagger\dagger} = -\eta$ and $\kappa^{\dagger\dagger} = -\kappa$, and that η and κ commute with their conjugate. Therefore, U_1 is pseudounitary as wanted. Moreover, the matrices η and κ must also fulfill (F.52). It is not difficult to check that this is the case if one writes

$$\eta = \begin{pmatrix} \eta_1 & 0 \\ 0 & -\eta_1^* \end{pmatrix}_{TR} \quad \text{and} \quad \kappa = \begin{pmatrix} \kappa_1 & 0 \\ 0 & -\kappa_1^* \end{pmatrix}_{TR}. \tag{F.54}$$

Let us now turn to U_2. This matrix must contain the two remaining real parameters. One starts by doubling the matrix u_2 found in (F.42). This yields

$$U_2 = \begin{pmatrix} u_2 & 0 \\ 0 & v_2 \end{pmatrix}_{RA}, \quad \text{with} \quad \begin{cases} u_2 = \begin{pmatrix} e^{i\Xi} & 0 \\ 0 & e^{i\Phi} \end{pmatrix}_{BF}, \\ v_2 = \mathbb{1}_{BF} \otimes \mathbb{1}_{TR} \end{cases} \tag{F.55}$$

where Ξ and Φ are 2×2 diagonal matrices. Then, it is easy to see that the time-reversal doubling requirement (F.52) forces

$$u_2 = \begin{pmatrix} e^{i\xi\sigma_3^{TR}} & 0 \\ 0 & e^{i\phi\sigma_3^{TR}} \end{pmatrix}_{BF}, \tag{F.56}$$

where the angles ξ and ϕ lie within $[0, 2\pi)$.

To complete the description of the Efetov's σ model with unitary symmetry using these coordinates, it just remains to be said what the measure dQ is.

The quite tedious calculation of this measure is performed in [34] and yields

$$dQ = J_1 dR_1 J_2 dR_2 d\hat{\theta}, \tag{F.57}$$

with the definitions

$$\begin{aligned} J_1 &= \tfrac{1}{2^8}\left(\cosh\theta_1 - \cos\theta\right)^{-2} & dR_1 &= d\eta_1 d\eta_1^* d\kappa_1^* d\kappa_1 \\ J_2 &= \tfrac{2^2}{\pi^2}\sin\theta \mathrm{sh}\theta_1 & dR_2 &= d\phi d\xi \\ d\hat{\theta} &= d\theta d\theta_1 \end{aligned} \tag{F.58}$$

F.2.3 Orthogonal Symmetry

The polar coordinates for the Efetov's σ model space with orthogonal symmetry are obtained in a way totally similar to the unitary case in section F.2.2. The only difference is that the Q matrix is not required to be diagonal in the TR space anymore, and one thus needs twice more real and anticommuting parameters. One starts with

$$Q = U Q_0 \bar{U}, \tag{F.59}$$

where Q_0 is diagonal in the Bose-Fermi space and carries the symmetries $\bar{Q}_0 = Q_0^{-1} = Q_0$. This job is achieved by

$$Q_0 = \begin{pmatrix} \cos\hat{\theta} & i\sin\hat{\theta} \\ -i\sin\hat{\theta} & -\cos\hat{\theta} \end{pmatrix}_{RA}, \quad \hat{\theta} = \begin{pmatrix} i\theta_B & 0 \\ 0 & \theta_F \end{pmatrix}_{BF}, \tag{F.60}$$

where θ_B and θ_F are 2×2 real symmetric matrices acting in the TR space. Let us write

$$\theta_B = \begin{pmatrix} \theta_1 & \theta_2 \\ \theta_2 & \theta_3 \end{pmatrix}_{TR}, \qquad \theta_F = \begin{pmatrix} \theta & \theta'' \\ \theta'' & \theta' \end{pmatrix}_{TR}. \tag{F.61}$$

The time-reversal requirement $Q_0^\tau = \sigma_3^{RA} Q_0 \sigma_3^{RA}$ reads

$$\begin{pmatrix} {Q_{0\downarrow\downarrow}}^T & \sigma_3^{BF}{Q_{0\uparrow\downarrow}}^T \\ {Q_{0\downarrow\uparrow}}^T\sigma_3^{BF} & {Q_{0\uparrow\uparrow}}^T \end{pmatrix}_{TR} = \sigma_3^{RA} \begin{pmatrix} Q_{0\uparrow\uparrow} & Q_{0\uparrow\downarrow} \\ Q_{0\downarrow\uparrow} & Q_{0\downarrow\downarrow} \end{pmatrix}_{TR} \sigma_3^{RA}. \tag{F.62}$$

Here, we used the fact that Q_0 is diagonal in Bose-Fermi space and thus all its components in time-reversal space commute with σ_3^{BF}. Since θ_B and θ_F are symmetric in TR space, Q_0 is also symmetric in this space, and hence, among the 4 equations in the components of (F.62), only two are independent. Moreover, these two equations decouple in Bose-Fermi space, and they can thus be written

$$Q_{0s\uparrow\uparrow}{}^{T} = \sigma_3^{RA} Q_{0s\downarrow\downarrow} \sigma_3^{RA}, \quad Q_{0s\uparrow\downarrow}{}^{T} = (-1)^s \sigma_3^{RA} Q_{0s\uparrow\downarrow} \sigma_3^{RA}, \tag{F.63}$$

for $s = B, F$ with the conventions $(-1)^B = 1$ and $(-1)^F = -1$. The two equations (F.63) take place in the retarded-advanced space. With (F.60) and (F.61), the first equation leads to $\theta_3 = \theta_1$ and $\theta' = \theta$. The second equation is trivially satisfied in the bosonic sector but forces $\theta'' = 0$ in the fermionic one. Hence, the angle matrices (F.61) become

$$\theta_B = \begin{pmatrix} \theta_1 & \theta_2 \\ \theta_2 & \theta_1 \end{pmatrix}_{TR}, \qquad \theta_F = \begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix}_{TR}, \tag{F.64}$$

with $\theta_1 \in [0, \infty)$ and $\theta \in [0, \pi]$, and they already contain 3 of the 8 real parameters we are after.

As in the previous section, the supermatrix U is taken pseudounitary, $\bar{U}U = 1$, and diagonal in the retarded-advanced space. It is factorized as the product of two pseudounitary matrices $U = U_1 U_2$, where U_1 diagonalizes Q in Bose-Fermi space and contains all the 8 anticommuting variables, and U_2 contains the remaining 5 real parameters. These two matrices are also required to satisfy the time-reversal doubling condition (F.47).

Let us first investigate U_1. In retarded-advanced space, one just has

$$U_1 = \begin{pmatrix} u_1 & 0 \\ 0 & v_1 \end{pmatrix}_{RA}, \tag{F.65}$$

where u_1 is unitary and v_1 is pseudounitary. The choice made in [34], that we

will follow, is

$$u_1 \equiv \exp\begin{pmatrix} 0 & -2\left(1-\frac{4}{3}\eta^\dagger\eta\right)\eta^\dagger \\ 2\eta\left(1-\frac{4}{3}\eta^\dagger\eta\right) & 0 \end{pmatrix}_{BF} \tag{F.66}$$

$$v_1 \equiv \exp\begin{pmatrix} 0 & -2i\left(1+\frac{4}{3}\kappa^\dagger\kappa\right)\kappa^\dagger \\ 2i\kappa\left(1+\frac{4}{3}\kappa^\dagger\kappa\right) & 0 \end{pmatrix}_{BF} \tag{F.67}$$

where η and κ are 2×2 matrices with anticommuting components. The peculiar choice for the supermatrices in the exponents will generate quite simple coefficients once the exponentials are performed. Notice that $\log u_1$ and $\log v_1$ are indeed pseudohermitian and hermitian respectively if $\eta^{\dagger\dagger} = -\eta$, $(\eta^\dagger\eta)^\dagger = \eta^\dagger\eta$, and similarly for κ. One also has to impose the time-reversal doubling requirement $U_1^\tau = \bar{U}_1$ as in (F.47), that is $u_1^\tau = u_1^\dagger$ and $v_1^\tau = \sigma_3^{BF} v_1^\dagger \sigma_3^{BF}$. These equations can also be written in terms of the exponents in (F.66) and (F.67) as

$$(\log u_1)^\tau = (\log u_1)^\dagger \quad \text{and} \quad (\log v_1)^\tau = \sigma_3^{BF}\,(\log u_1)^\dagger\,\sigma_3^{BF}. \tag{F.68}$$

It is not difficult to check that the generalizations

$$\eta \equiv \begin{pmatrix} \eta_1^* & \eta_2 \\ \eta_2^* & \eta_1 \end{pmatrix}_{TR} \quad \text{and} \quad \kappa \equiv \begin{pmatrix} \kappa_1^* & \kappa_2 \\ \kappa_2^* & \kappa_1 \end{pmatrix}_{TR} \tag{F.69}$$

of (F.54) fulfill all the above conditions. In particular,

$$\eta^\dagger\eta = (\eta_1^*\eta_1 + \eta_2^*\eta_2)\,\mathbb{1}_{TR} \quad \text{and} \quad \kappa^\dagger\kappa = (\kappa_1^*\kappa_1 + \kappa_2^*\kappa_2)\,\mathbb{1}_{TR}. \tag{F.70}$$

Performing the exponentials in (F.66) and (F.67) yields

$$u_1 = \begin{pmatrix} 1-2\eta^\dagger\eta+6(\eta^\dagger\eta)^2 & -2(1-2\eta^\dagger\eta)\eta^\dagger \\ 2\eta(1-2\eta^\dagger\eta) & 1-2\eta\eta^\dagger+6(\eta\eta^\dagger)^2 \end{pmatrix}_{BF} \tag{F.71}$$

$$v_1 = \begin{pmatrix} 1+2\kappa^\dagger\kappa+6(\kappa^\dagger\kappa)^2 & -2i(1+2\kappa^\dagger\kappa)\kappa^\dagger \\ 2i\kappa(1+2\kappa^\dagger\kappa) & 1+2\kappa\kappa^\dagger+6(\kappa\kappa^\dagger)^2 \end{pmatrix}_{BF} \tag{F.72}$$

Let us now turn to the supermatrix U_2. This matrix is diagonal in retarded-

advanced space and in Bose-Fermi space, so that one can write as previously

$$U_2 = \begin{pmatrix} u_2 & 0 \\ 0 & v_2 \end{pmatrix}_{RA} , \quad \text{with} \quad \begin{cases} u_2 = \begin{pmatrix} u_{2B} & 0 \\ 0 & u_{2F} \end{pmatrix}_{BF} \\ v_2 = \begin{pmatrix} v_{2B} & 0 \\ 0 & v_{2F} \end{pmatrix}_{BF} \end{cases} \tag{F.73}$$

It must contain the 5 remaining real parameters and satisfy the time-reversal doubling and the pseudounitarity conditions, that is

$$\begin{aligned} u_2^\tau = u_2^\dagger \quad , \quad v_2^\tau = v_2^\dagger \quad , \\ u_2^\dagger u_2 = 1 \quad , \quad \bar{v}_2 v_2 = v_2^\dagger v_2 = 1. \end{aligned} \tag{F.74}$$

The first condition on u_2 implies that the blocks u_{2B} and u_{2F} must be of the form

$$u_{2B} = \begin{pmatrix} u_{2B\uparrow\uparrow} & u^*_{2B\downarrow\uparrow} \\ u_{2B\downarrow\uparrow} & u^*_{2B\uparrow\uparrow} \end{pmatrix}_{TR} , \quad u_{2F} = \begin{pmatrix} u_{2F\uparrow\uparrow} & u^*_{2F\downarrow\uparrow} \\ -u_{2F\downarrow\uparrow} & u^*_{2F\uparrow\uparrow} \end{pmatrix}_{TR} . \tag{F.75}$$

The difference of sign in the component (2,1) of these matrices has a big impact. Indeed, once the pseudounitary condition in (F.74) are imposed to u_{2B} and u_{2F} in (F.75), u_{2B} is found to be diagonal, whereas u_{2F} just needs to have determinant unity. In other words

$$u_{2B} = e^{i\xi\sigma_3^{TR}} \quad \text{and} \quad u_{2F} = \begin{pmatrix} a & b^* \\ -b & a^* \end{pmatrix}_{TR} , \tag{F.76}$$

where $\xi \in [0, 2\pi)$ and $a, b \in \mathbb{C}$ are such that $|a|^2 + |b|^2 = 1$. Notice that the set of such matrices u_{2F} is precisely SU(2) which is of real dimension 3. We parametrize the matrix u_{2F} in SU(2) as in [34]. First, it is easy to convince oneself that any matrix A in U(2) can be written

$$A = (1 - iM)(1 + iM)^{-1} \quad \text{for} \quad M = \begin{pmatrix} m & m_1^* \\ m_1 & m' \end{pmatrix} \tag{F.77}$$

some hermitian matrix. The four parameters m, m', $\Re m_1$ and $\Im m_1$ can be seen as parameters for $U(2)$. Then, it can be checked that A as determinant unity if and only if $m' = -m$. Therefore, u_{2F} can be written

$$u_{2F} = (1 - iM)(1 + iM)^{-1} \quad \text{with} \quad M = \begin{pmatrix} m & m_1^* \\ m_1 & -m \end{pmatrix}_{TR}, \tag{F.78}$$

for $m \in \mathbb{R}$ and $m_1 \in \mathbb{C}$. Hence, altogether u_2 contains 4 real parameters. One real parameter is still missing and has thus to be contained in v_2. Since by (F.74) v_2 has to satisfy exactly the same conditions as u_2, it is natural to choose

$$v_{2B} = e^{i\chi\sigma_3^{TR}} \quad \text{and} \quad v_{2F} = \mathbb{1}_{TR} \tag{F.79}$$

for $\chi \in [0, 2\pi)$. This ends the description of the Efetov's σ model space with orthogonal symmetry.

Finally, the invariant measure dQ introduced in (F.6) in these polar coordinates is found to be

$$dQ = J_1 dR_1 J_2 dR_2 d\hat{\theta}, \tag{F.80}$$

where

$$\begin{aligned} J_1 &= \frac{1}{2^{20}} \big(\mathrm{ch}(\theta_1 + \theta_2) - \cos\theta\big)^{-2} \big(\mathrm{ch}(\theta_1 - \theta_2) - \cos\theta\big)^{-2} \\ dR_1 &= d\eta_1 d\eta_1^* d\eta_2 d\eta_2^* d\kappa_1^* d\kappa_1 d\kappa_2^* d\kappa_2 \\ J_2 &= \frac{2^{12}}{\pi^4} \frac{\sin^3\theta \, \mathrm{sh}\theta_1 \, \mathrm{sh}\theta_2}{(1 + m^2 + |m_1|^2)^3} \\ dR_2 &= dm dm_1 dm_1^* d\phi d\chi \\ d\hat{\theta} &= d\theta d\theta_1 d\theta_2. \end{aligned} \tag{F.81}$$

The reader interested in the derivation of this formula can refer to [34].

F.3 The Mean Field Integrals

In the mean field theory exposed in Chapter 4, the integral $I_\pi(t, \sigma)$ in (4.33), which provides the autocorrelation functions, can be performed exactly. The

purpose of this appendix is to calculate this mean field integral in the unitary symmetry case, making use of the polar coordinates introduced in F.2.2.

In Theorem 4.1 the mean field source-free action S_0^{MF} is defined. It is direct to check that, in the case of unitary symmetry, this action can be written in terms of the polar coordinates introduced in F.2.2 as

$$S_0^{MF} = 2B\epsilon\Big(\cosh\theta_1 - \cos\theta\Big). \tag{F.82}$$

It is convenient to work with the new variables

$$\lambda_1 \equiv \cosh\theta_1 \in [1,\infty) \quad \text{and} \quad \lambda = \cos\theta \in [-1,1] \tag{F.83}$$

instead of θ_1 and θ. In the unitary case, the mean field integrals for the autocorrelation functions are always of the type

$$I_n(\epsilon) \equiv \int dQ\; e^{-S_0^{\mathrm{MF}}} f_n(Q) \tag{F.84}$$

$$= \frac{1}{2^6\pi^2}\int_1^\infty d\lambda_1 \int_{-1}^1 d\lambda \int dR_1 dR_2 \frac{e^{-2B\epsilon(\lambda_1-\lambda)}}{(\lambda_1-\lambda)^2} f_n(Q) \tag{F.85}$$

The index n stands for an integer $n \geq 2$. In these last two formulae, $f_n(Q)$ is a function of the 8 parameters describing Q. It satisfies

$$f_n(Q) = \hat{f}_n(\lambda_1,\lambda,\phi,\xi)\cdot\eta_1^*\eta_1\kappa_1^*\kappa_1 + r_n(Q), \tag{F.86}$$

where $r_n(Q)$ is of degree inferior to 4 in the anticommuting variables, and

$$\lim_{\lambda_1\to\infty}\frac{\hat{f}_n(\lambda_1,\lambda,\phi,\xi)}{\lambda_1^n} = c, \quad c\in\mathbb{R}^*. \tag{F.87}$$

In practice, this function corresponds to some derivatives of the supersymmetry breaking factor $P_{[\alpha]}(j)$ evaluated at vanishing sources. In (F.85), the measures dR_1 and dR_2 are those defined in (F.58), and the ratio in the integrand is made from the exponential of $-S_0^{MF}$ and the factor J_1 of the Berezinian (F.58). Notice that the functions $\sin\theta$ and $\sinh\theta_1$ in the part J_2 of this Berezinian are absorbed by the change of variables (F.83), and the numerical factor in front of the superintegral (F.85) comes from the numerical factors in J_1 and J_2.

The unitary mean field integrals (F.84) are singular as $\epsilon \longrightarrow 0$, and the most singular term yields an autocorrelation function. It will be seen in the sequel that, with $f_n(Q)$ of the form (F.86),

$$\lim_{\epsilon \to 0} \epsilon^{n-1} I_n(\epsilon) = -\frac{c}{2^3} \frac{(n-2)!}{(2B)^{n-1}}, \tag{F.88}$$

which is positive and finite.

In fact, the explicit calculations of the unitary mean field integrals show that $f_n(Q)$ does not depend on the angles ϕ and ξ originating from the pseudounitary matrix U_2 in (F.55). Hence, the integral of the differential dR_2 merely yields a factor $(2\pi)^2$, and (F.85) becomes

$$I_n(\epsilon) = \frac{1}{2^4} \int_1^\infty d\lambda_1 \int_{-1}^1 d\lambda \int dR_1 \frac{e^{-2B\epsilon(\lambda_1 - \lambda)}}{(\lambda_1 - \lambda)^2} f_n(Q). \tag{F.89}$$

However, the following arguments are independent of this remark. In particular, the result (F.88) still holds if $f_n(Q)$ depends on ϕ and ξ. But for the sake of simplicity, we will subsequently work with (F.89).

Let us start with an important remark. The function f_n in (F.84) is a polynomial in the four Grassmann variables contained in U_1. If one of the monomial does not involve all the four anticommuting variables, its Berezin integral over these anticommuting variables vanishes. But on the other hand, the integrand in (F.89) coming from this monomial may have a non-integrable singularity at $\lambda_1 = \lambda = 1$ so that, if the integration of the variables λ_1 and λ is performed first, one gets a divergence. In order to find out what really becomes of such a monomial, one considers the derivatives

$$I_n''(\epsilon) = \frac{B^2}{4} \int_1^\infty d\lambda_1 \int_{-1}^1 d\lambda \int dR_1 e^{-2B\epsilon(\lambda_1 - \lambda)} f_n(Q). \tag{F.90}$$

This procedure kills the singularity at $\lambda_1 = \lambda = 1$ in the integrand, and hence, it removes the previous ambiguity. Now, a monomial of f_n that does not involve all the four anticommuting variables cannot contribute to $I_n''(\epsilon)$. This motivates the following definition. For f and f' two polynomials in the four Grassmann variables, one writes

$$f \overset{1}{\sim} f' \tag{F.91}$$

whenever the monomials in f and f' containing all the Grassmann variables coincide. Two polynomials equivalent under this relation thus yield the same function $I''(\epsilon)$ in (F.90). Besides, $I_n(\epsilon)$ can be recovered from $I_n''(\epsilon)$ by integration. Indeed, for any $a > 0$,

$$I_n(\epsilon) = \int_a^{\epsilon} \int_a^{\epsilon_1} I_n''(\epsilon_2)\, d\epsilon_2\, d\epsilon_1 + (\epsilon - a)I'(a) + I(a). \tag{F.92}$$

However, only the first term in the Laurent series of $I_n(\epsilon)$ is relevant. Since a singular contribution will indeed be found in $I_n(\epsilon)$, the last two terms in the right-hand side of (F.92), which are analytic, can already be dropped. One deduces that two functions $f_n(Q)$ and $f_n'(Q)$ that are equivalent under (F.91) lead to the same mean field integral (F.84) up to analytic terms in ϵ. In other words, the function $r_n(Q)$ defined in (F.86) is irrelevant for our purposes.

There is another important general observation that can be made about the asymptotics $\epsilon \longrightarrow 0$ of the unitary mean field integrals (F.84). By the preceding remark, $I_n(\epsilon)$ in (F.89) can be written

$$I_n(\epsilon) = -\frac{1}{2^4} \int_1^{\infty} d\lambda_1 \int_{-1}^{1} d\lambda\, \frac{e^{-2B\epsilon(\lambda_1 - \lambda)}}{(\lambda_1 - \lambda)^2} \hat{f}_n(\lambda_1, \lambda), \tag{F.93}$$

where $\hat{f}_n(\lambda_1, \lambda)$ is given in (F.86). Once again, the forthcoming argument still holds if f_n, and in particular $\hat{f}_n$, depends of the angles ϕ and χ. In practice, the function $\hat{f}_n$ always involves a factor killing the non-integrable singularity at $\lambda_1 = \lambda = 1$. In fact, the integrand of (F.93) can even be continued at this point. The singularity of $I_n(\epsilon)$ at the origin comes from the region of asymptotically large λ_1. In order to see this, let us rescale

$$\lambda_1 \mapsto \lambda_1' = \epsilon\lambda_1. \tag{F.94}$$

Multiplying $I_n(\epsilon)$ in (F.93) by the regularizing factor ϵ^{n-1}, and then taking the limit of vanishing ϵ provides

$$\lim_{\epsilon \to 0} \epsilon^{n-1} I(\epsilon) = -\frac{1}{2^4} \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} d\lambda_1' \int_{-1}^{1} d\lambda\, \frac{e^{-2B\lambda_1'} e^{2B\epsilon\lambda}}{(\lambda_1' - \epsilon\lambda)^2} \epsilon^n \hat{f}_n\left(\frac{\lambda_1'}{\epsilon}, \lambda\right). \tag{F.95}$$

Taking the limit $\epsilon \to 0$ in the integration domain of λ_1 and in the integrand, and making use of the asymptotic behavior (F.87) shows that

$$\begin{aligned} \lim_{\epsilon \to 0} \epsilon^{n-1} I(\epsilon) &= -\frac{c}{2^4} \int_0^\infty d\lambda_1' \int_{-1}^1 d\lambda \; \lambda_1'^{n-2} e^{-2B\lambda_1'} \\ &= -\frac{c}{2^3} \frac{(n-2)!}{(2B)^{n-1}}, \end{aligned} \tag{F.96}$$

which is precisely the result announced in (F.88).

In fact, when dealing with a unitary mean field integral (F.84) , it is more convenient to make use of the two preceding remarks in the reverse order, and one first isolates in the function $f_n(Q)$ the leading terms as $\lambda_1 \to \infty$. It is then convenient to introduce the equivalence relation

$$f_1(Q) \overset{2}{\sim} f_2(Q) \quad \text{iff} \quad \lim_{\lambda_1 \to \infty} \frac{f_1(Q)}{f_2(Q)} = 1. \tag{F.97}$$

This notation means that $f_1(Q)$ and $f_2(Q)$, seen as polynomials in the 4 anti-commuting variables, become identical in the limit of large λ_1. This relation is stable under multiplication, that is, if $f_1 \overset{2}{\sim} f_2$ and $g_1 \overset{2}{\sim} g_2$, then $f_1 g_1 \overset{2}{\sim} f_2 g_2$. The relation $\overset{1}{\sim}$ introduced in (F.91) does not have this property. In practice, the function $r_n(Q)$ defined in (F.86) always has the property

$$r_n(Q) \overset{2}{\sim} \lambda_1^n \cdot g(\lambda, \eta_1^*, \eta_1, \kappa_1^*, \kappa_1) \tag{F.98}$$

for some non-zero polynomial g of degree inferior to 4, namely, $r_n(Q)$ is exactly of same order in λ_1 as $\hat{f}_n(Q)$. Hence,

$$f_n(Q) \overset{2}{\sim} \lambda_1^n \Big(c \cdot \eta_1^* \eta_1 \kappa_1^* \kappa_1 + g(\lambda, \eta_1^*, \eta_1, \kappa_1^*, \kappa_1) \Big). \tag{F.99}$$

Now, only the first term in (F.99), which involves all the four anticommuting variables, contributes to the singular terms of $I_n(\epsilon)$. We therefore introduce a new equivalence relation between the functions $f(Q)$ of the type (F.99) and write

$$f_1(Q) \sim f_2(Q) \quad \text{iff} \quad c_1 = c_2, \tag{F.100}$$

where c_1 and c_2 refer to the real numbers c in (F.99) for $f_1(Q)$ and $f_2(Q)$ respectively. Two functions $f_1(Q)$ and $f_2(Q)$ equivalent under this relation yields the same result (F.88).

F.4 Proof of Property 4.2

We show that $\xi^{\mathrm{MF}}_{[\alpha]}(0, j_r) = 1$ for all $j_r \in \mathbb{R}$ in a sufficiently small neighborhood of the origin. The other cases treated in Property 4.2 can then be obtained after some simple generalizations of the argument shown here. Let us first write the retarded source j_r as

$$j_r \equiv 1 + \delta. \tag{F.101}$$

The mean field generating function at $j_a = 0$ becomes

$$\xi^{\mathrm{MF}}_{\beta}(0, j_r) = \int dQ\; e^{-S^{\mathrm{MF}}_0} \det\left[1 + \frac{1+\delta}{2}(Q_{BB;rr} - 1)\right]^{-\frac{1}{2}}. \tag{F.102}$$

The fact that the supersymmetry breaking factor in the integrand only involves one source parameter δ enables to define a change of variable $Q \mapsto X$ of unit Berezinian leaving the mean field source free action unchanged and such that δ disappears from this supersymmetry breaking factor. Indeed, if one sets

$$Q \mapsto X \equiv \begin{pmatrix} 1+\delta & 0 \\ 0 & 1-\delta \end{pmatrix}_{RA} Q + \begin{pmatrix} -\delta & 0 \\ 0 & \delta \end{pmatrix}_{RA}, \tag{F.103}$$

the mean field source free action satisfies

$$S^{\mathrm{MF}}_0 \equiv \mathrm{str}\sigma_3^{RA} Q = \mathrm{str}\sigma_3^{RA} X \tag{F.104}$$

up to terms of higher degrees in ϵ. To get this equality, we used $\mathrm{str}Q_{aa} = -\mathrm{str}Q_{rr}$, which can be seen on (4.6) and (4.7) for example. Moreover, $\xi^{\mathrm{MF}}_{[\alpha]}(0, j_r)$ can be written as,

$$\xi^{\mathrm{MF}}_{[\alpha]}(0, j_r) = \int dX\; e^{-S^{\mathrm{MF}}_0} \det\left[1 + \frac{1}{2}(X_{BB;rr} - 1)\right]^{-\frac{1}{2}}. \tag{F.105}$$

The Berezinian of (F.103) is indeed one since the measure dQ is invariant under a shift. The integration domain is not invariant under (F.103), but it can be deformed back without crossing any singularity.

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